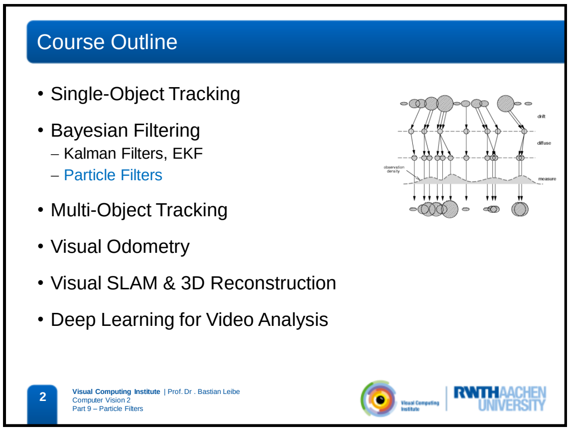
Computer Vision 2
WS 2018/19

Part 9 – Particle Filters
21.11.2018

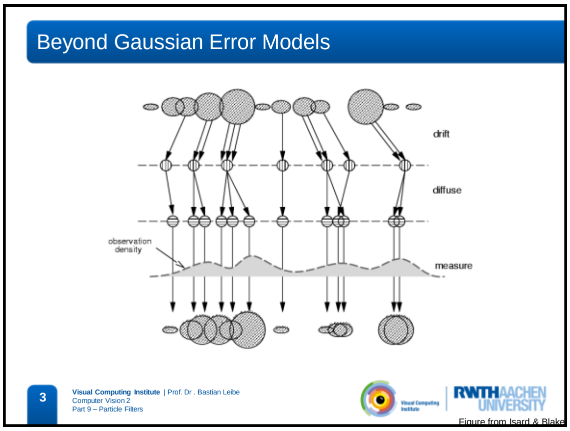
Prof. Dr. Bastian Leibe

RWTH Aachen University, Computer Vision Group
http://www.vision.rwth-aachen.de



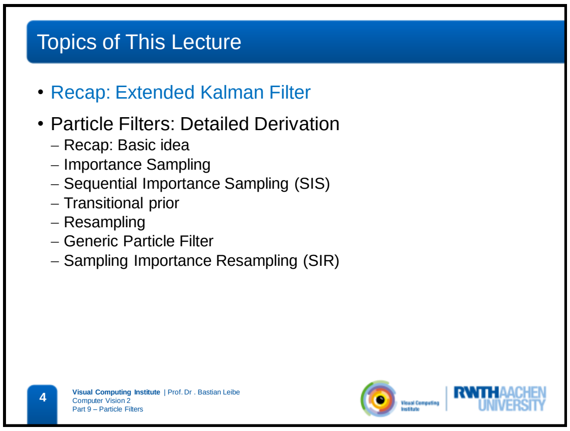
Course Outline

- Single-Object Tracking
- Bayesian Filtering
  - Kalman Filters, EKF
  - Particle Filters
- Multi-Object Tracking
- Visual Odometry
- Visual SLAM & 3D Reconstruction
- Deep Learning for Video Analysis



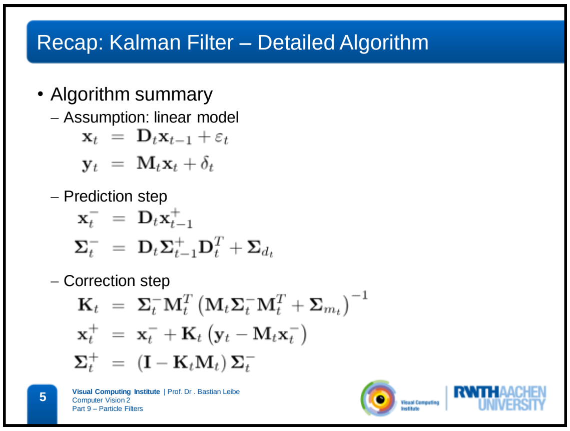
Beyond Gaussian Error Models

Figure from Isard & Blake



Topics of This Lecture

- Recap: Extended Kalman Filter
- Particle Filters: Detailed Derivation
  - Recap: Basic idea
  - Importance Sampling
  - Sequential Importance Sampling (SIS)
  - Transitional prior
  - Resampling
  - Generic Particle Filter
  - Sampling Importance Resampling (SIR)



Recap: Kalman Filter – Detailed Algorithm

- Algorithm summary
  - Assumption: linear model

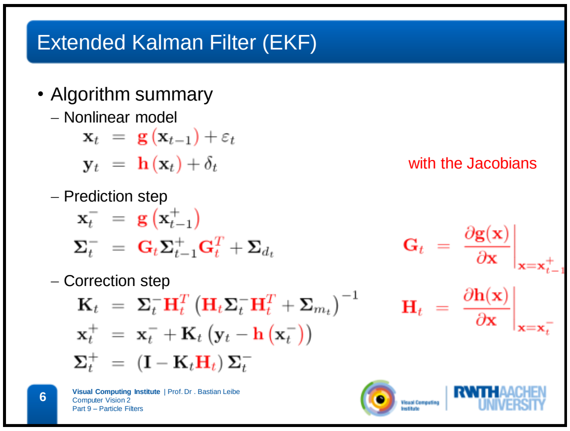
$$\begin{aligned}\mathbf{x}_t &= \mathbf{D}_t\mathbf{x}_{t-1} + \varepsilon_t \\ \mathbf{y}_t &= \mathbf{M}_t\mathbf{x}_t + \delta_t\end{aligned}$$

  - Prediction step

$$\begin{aligned}\mathbf{x}_t^- &= \mathbf{D}_t\mathbf{x}_{t-1}^+ \\ \mathbf{\Sigma}_t^- &= \mathbf{D}_t\mathbf{\Sigma}_{t-1}^+\mathbf{D}_t^T + \mathbf{\Sigma}_{d_t}\end{aligned}$$

  - Correction step

$$\begin{aligned}\mathbf{K}_t &= \mathbf{\Sigma}_t^-\mathbf{M}_t^T\left(\mathbf{M}_t\mathbf{\Sigma}_t^-\mathbf{M}_t^T + \mathbf{\Sigma}_{m_t}\right)^{-1} \\ \mathbf{x}_t^+ &= \mathbf{x}_t^- + \mathbf{K}_t\left(\mathbf{y}_t - \mathbf{M}_t\mathbf{x}_t^-\right) \\ \mathbf{\Sigma}_t^+ &= \left(\mathbf{I} - \mathbf{K}_t\mathbf{M}_t\right)\mathbf{\Sigma}_t^-\end{aligned}$$



Extended Kalman Filter (EKF)

- Algorithm summary
  - Nonlinear model

$$\begin{aligned}\mathbf{x}_t &= \mathbf{g}\left(\mathbf{x}_{t-1}\right) + \varepsilon_t \\ \mathbf{y}_t &= \mathbf{h}\left(\mathbf{x}_t\right) + \delta_t\end{aligned}$$

with the Jacobians

  - Prediction step

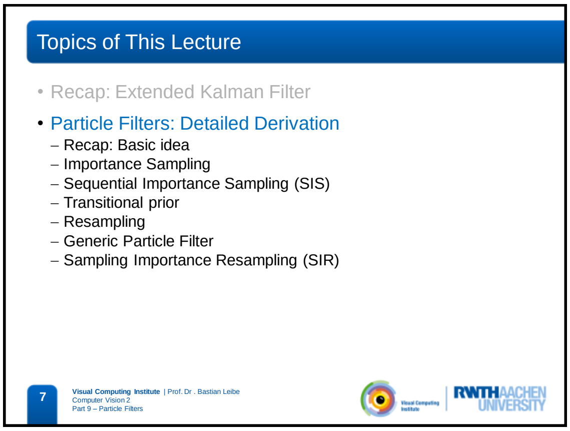
$$\begin{aligned}\mathbf{x}_t^- &= \mathbf{g}\left(\mathbf{x}_{t-1}^+\right) \\ \mathbf{\Sigma}_t^- &= \mathbf{G}_t\mathbf{\Sigma}_{t-1}^+\mathbf{G}_t^T + \mathbf{\Sigma}_{d_t}\end{aligned}$$

$$\mathbf{G}_t = \left.\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{t-1}^+}$$

  - Correction step

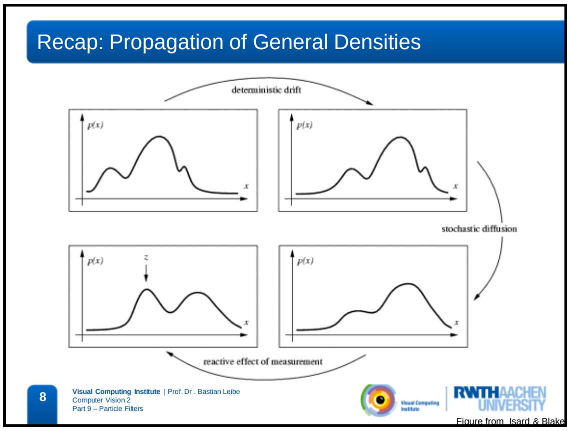
$$\begin{aligned}\mathbf{K}_t &= \mathbf{\Sigma}_t^-\mathbf{H}_t^T\left(\mathbf{H}_t\mathbf{\Sigma}_t^-\mathbf{H}_t^T + \mathbf{\Sigma}_{m_t}\right)^{-1} \\ \mathbf{x}_t^+ &= \mathbf{x}_t^- + \mathbf{K}_t\left(\mathbf{y}_t - \mathbf{h}\left(\mathbf{x}_t^-\right)\right) \\ \mathbf{\Sigma}_t^+ &= \left(\mathbf{I} - \mathbf{K}_t\mathbf{H}_t\right)\mathbf{\Sigma}_t^-\end{aligned}$$

$$\mathbf{H}_t = \left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_t^-}$$



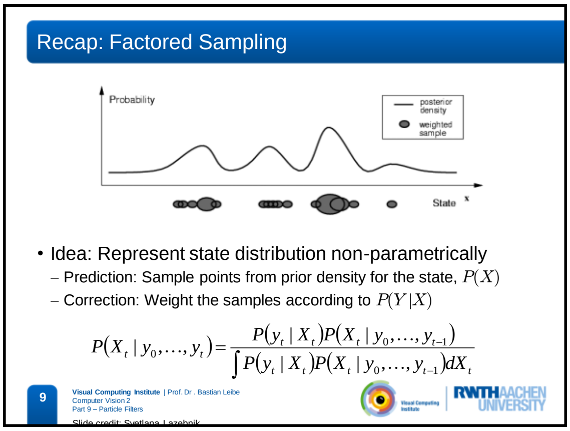
Topics of This Lecture

- Recap: Extended Kalman Filter
- Particle Filters: Detailed Derivation
  - Recap: Basic idea
  - Importance Sampling
  - Sequential Importance Sampling (SIS)
  - Transitional prior
  - Resampling
  - Generic Particle Filter
  - Sampling Importance Resampling (SIR)



Recap: Propagation of General Densities

Figure from Isard & Blake

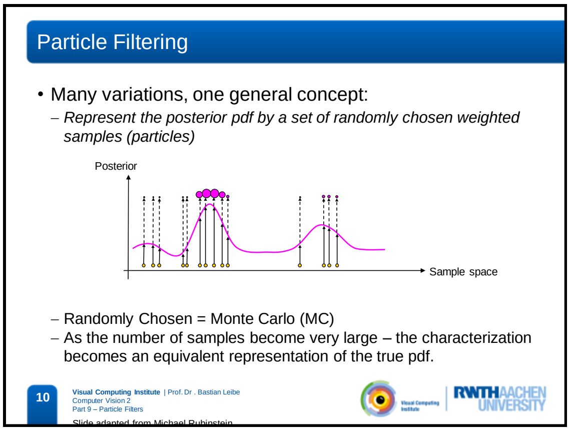


Recap: Factored Sampling

- Idea: Represent state distribution non-parametrically
  - Prediction: Sample points from prior density for the state, $P(X)$
  - Correction: Weight the samples according to $P(Y|X)$

$$P(X_t \mid y_0,\dots,y_t) = \frac{P(y_t \mid X_t)P(X_t \mid y_0,\dots,y_{t-1})}{\int P(y_t \mid X_t)P(X_t \mid y_0,\dots,y_{t-1})dX_t}$$

Slide credit: Svetlana Lazebnik

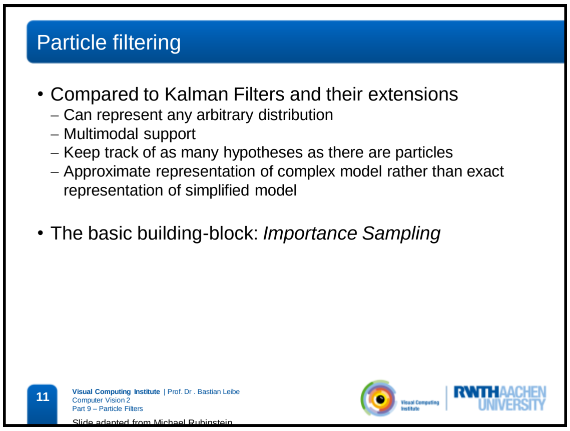


Particle Filtering

- Many variations, one general concept:
  - *Represent the posterior pdf by a set of randomly chosen weighted samples (particles)*

  - Randomly Chosen = Monte Carlo (MC)
  - As the number of samples become very large – the characterization becomes an equivalent representation of the true pdf.

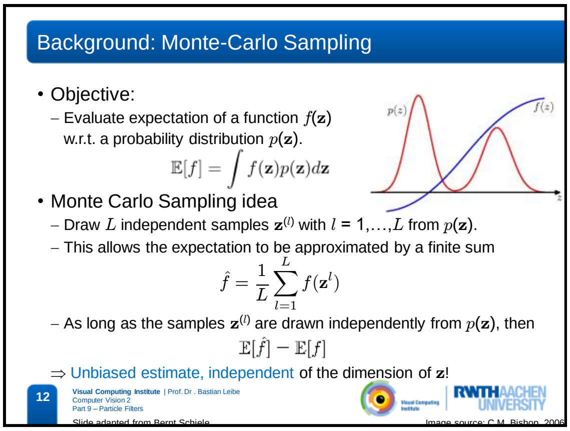
Slide adapted from Michael Rubinstein



Particle filtering

- Compared to Kalman Filters and their extensions
  - Can represent any arbitrary distribution
  - Multimodal support
  - Keep track of as many hypotheses as there are particles
  - Approximate representation of complex model rather than exact representation of simplified model
- The basic building-block: *Importance Sampling*

Slide adapted from Michael Rubinstein



Background: Monte-Carlo Sampling

- Objective:
  - Evaluate expectation of a function $f(\mathbf{z})$ w.r.t. a probability distribution $p(\mathbf{z})$.

$$\mathbb{E}[f] = \int f(\mathbf{z})p(\mathbf{z})d\mathbf{z}$$

- Monte Carlo Sampling idea
  - Draw L independent samples $\mathbf{z}^{(l)}$ with l = 1,…,L from $p(\mathbf{z})$.
  - This allows the expectation to be approximated by a finite sum

$$\hat{f} = \frac{1}{L}\sum_{l=1}^{L} f(\mathbf{z}^{l})$$

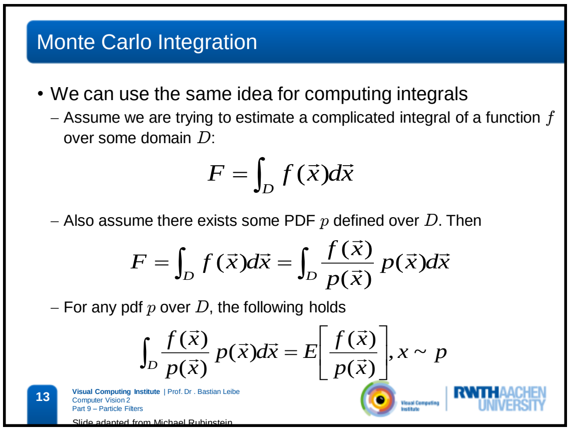
  - As long as the samples $\mathbf{z}^{(l)}$ are drawn independently from $p(\mathbf{z})$, then

$$\mathbb{E}[\hat{f}] - \mathbb{E}[f]$$

⇒ Unbiased estimate, independent of the dimension of $\mathbf{z}$!

Slide adapted from Bernt Schiele

Image source: C.M. Bishop, 2006



Monte Carlo Integration

- We can use the same idea for computing integrals
  - Assume we are trying to estimate a complicated integral of a function f over some domain D:

$$F = \int_D f(\vec{x})d\vec{x}$$

  - Also assume there exists some PDF p defined over D. Then

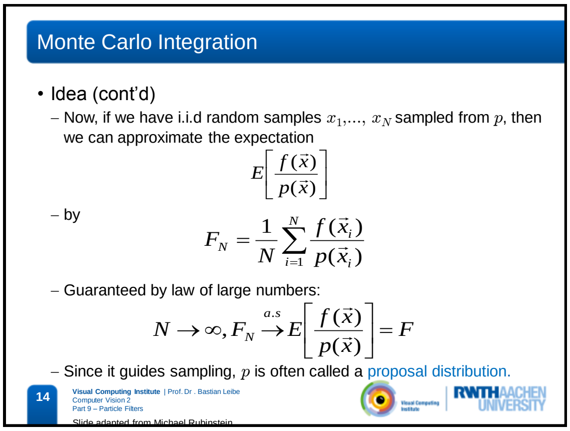
$$F = \int_D f(\vec{x})d\vec{x} = \int_D \frac{f(\vec{x})}{p(\vec{x})} p(\vec{x})d\vec{x}$$

  - For any pdf p over D, the following holds

$$\int_D \frac{f(\vec{x})}{p(\vec{x})} p(\vec{x})d\vec{x} = E\left[\frac{f(\vec{x})}{p(\vec{x})}\right], x \sim p$$

13

Visual Computing Institute | Prof. Dr . Bastian Leibe
Computer Vision 2
Part 9 – Particle Filters
Slide adapted from Michael Rubinstein



Monte Carlo Integration

- Idea (cont'd)
  - Now, if we have i.i.d random samples $x_1,..., x_N$ sampled from p, then we can approximate the expectation

$$E\left[\frac{f(\vec{x})}{p(\vec{x})}\right]$$

  - by

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(\vec{x}_i)}{p(\vec{x}_i)}$$

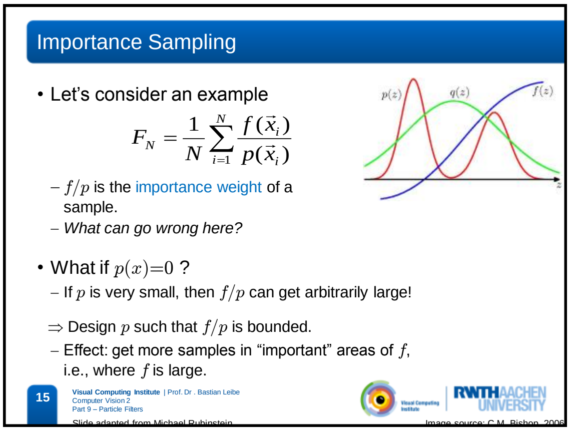
  - Guaranteed by law of large numbers:

$$N \to \infty, F_N \xrightarrow{a.s} E\left[\frac{f(\vec{x})}{p(\vec{x})}\right] = F$$

  - Since it guides sampling, p is often called a proposal distribution.

14

Visual Computing Institute | Prof. Dr . Bastian Leibe
Computer Vision 2
Part 9 – Particle Filters
Slide adapted from Michael Rubinstein



Importance Sampling

- Let's consider an example

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(\vec{x}_i)}{p(\vec{x}_i)}$$

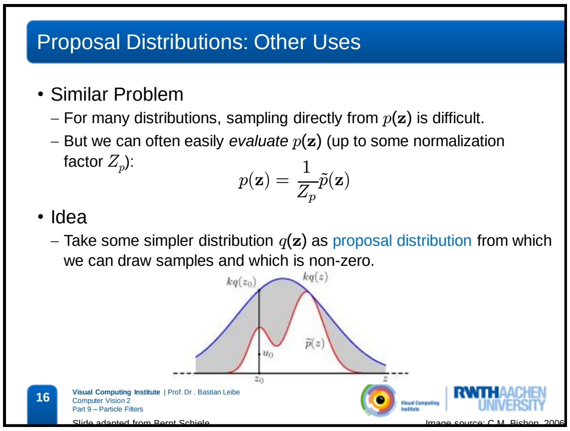
  - f/p is the importance weight of a sample.
  - *What can go wrong here?*
- What if $p(x)$=0 ?
  - If p is very small, then f/p can get arbitrarily large!

$\Rightarrow$ Design p such that f/p is bounded.

  - Effect: get more samples in "important" areas of f, i.e., where f is large.

15

Visual Computing Institute | Prof. Dr . Bastian Leibe
Computer Vision 2
Part 9 – Particle Filters
Slide adapted from Michael Rubinstein
Image source: C.M. Bishop, 2006



Proposal Distributions: Other Uses

- Similar Problem
  - For many distributions, sampling directly from $p(\mathbf{z})$ is difficult.
  - But we can often easily *evaluate* $p(\mathbf{z})$ (up to some normalization factor Z_p):

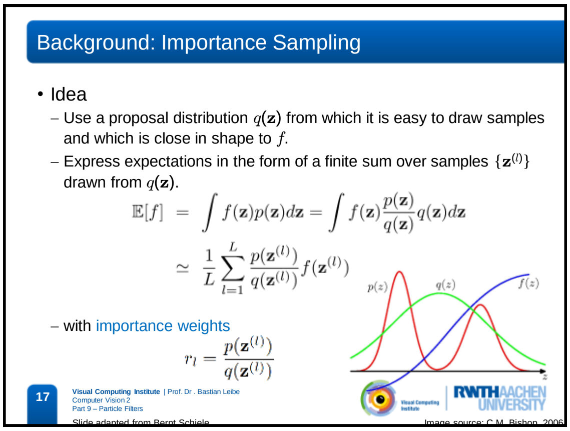
$$p(\mathbf{z}) = \frac{1}{Z_p}\tilde{p}(\mathbf{z})$$

- Idea
  - Take some simpler distribution $q(\mathbf{z})$ as proposal distribution from which we can draw samples and which is non-zero.

16

Visual Computing Institute | Prof. Dr . Bastian Leibe
Computer Vision 2
Part 9 – Particle Filters
Slide adapted from Bernt Schiele
Image source: C.M. Bishop, 2006

$kq(z_0)$
$kq(z)$
u_0
$\tilde{p}(z)$
z_0
z



Background: Importance Sampling

- Idea
  - Use a proposal distribution $q(\mathbf{z})$ from which it is easy to draw samples and which is close in shape to f.
  - Express expectations in the form of a finite sum over samples $\{\mathbf{z}^{(l)}\}$ drawn from $q(\mathbf{z})$.

$$\mathbb{E}[f] = \int f(\mathbf{z})p(\mathbf{z})d\mathbf{z} = \int f(\mathbf{z})\frac{p(\mathbf{z})}{q(\mathbf{z})}q(\mathbf{z})d\mathbf{z}$$

$$\simeq \frac{1}{L}\sum_{l=1}^{L}\frac{p(\mathbf{z}^{(l)})}{q(\mathbf{z}^{(l)})}f(\mathbf{z}^{(l)})$$

  - with importance weights

$$r_l = \frac{p(\mathbf{z}^{(l)})}{q(\mathbf{z}^{(l)})}$$

17

Visual Computing Institute | Prof. Dr . Bastian Leibe
Computer Vision 2
Part 9 – Particle Filters
Slide adapted from Bernt Schiele
Image source: C.M. Bishop, 2006

$p(z)$
$q(z)$
$f(z)$
z

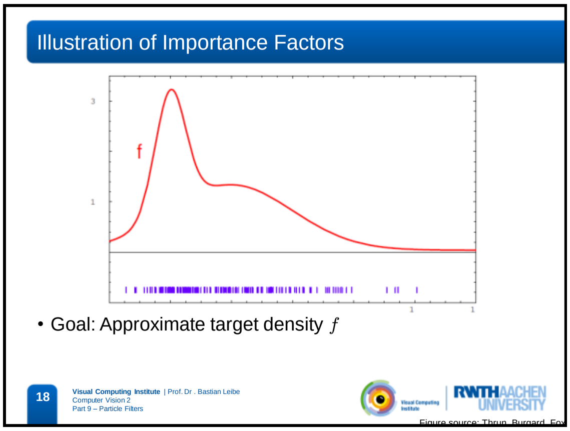


Illustration of Importance Factors

- Goal: Approximate target density f

18

Visual Computing Institute | Prof. Dr . Bastian Leibe
Computer Vision 2
Part 9 – Particle Filters
Figure source: Thrun, Burgard, Fox

3
1
f
1
1

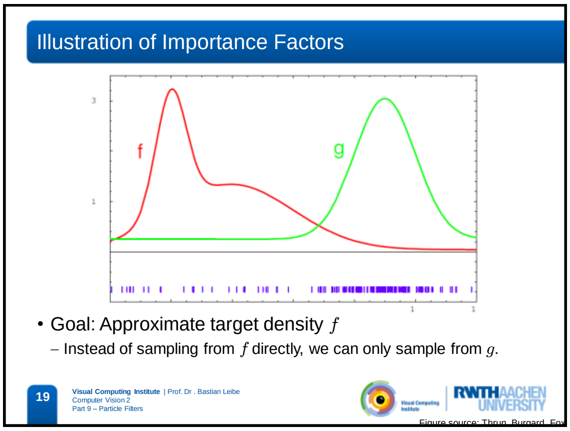


Illustration of Importance Factors

- Goal: Approximate target density f
  - Instead of sampling from f directly, we can only sample from g.

Figure source: Thrun, Burgard, Fox

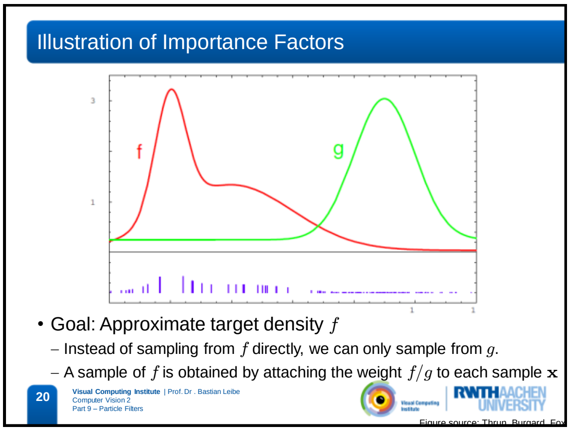


Illustration of Importance Factors

- Goal: Approximate target density f
  - Instead of sampling from f directly, we can only sample from g.
  - A sample of f is obtained by attaching the weight f/g to each sample $\mathbf{x}$

Figure source: Thrun, Burgard, Fox

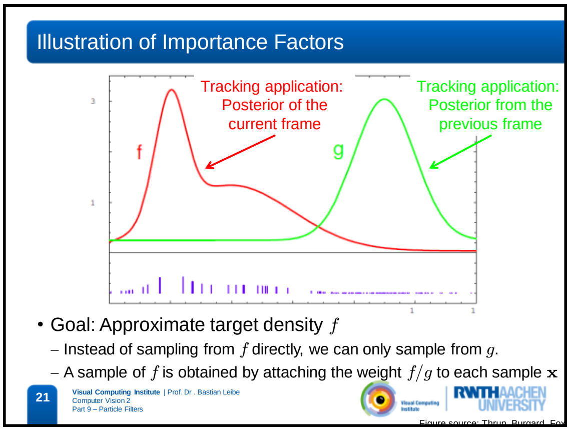


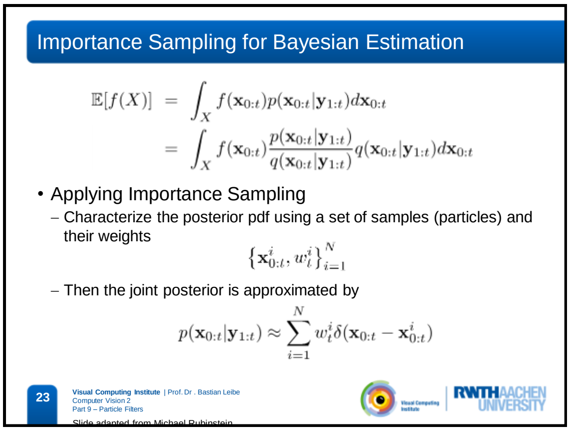
Illustration of Importance Factors

Tracking application: Posterior of the current frame

Tracking application: Posterior from the previous frame

- Goal: Approximate target density f
  - Instead of sampling from f directly, we can only sample from g.
  - A sample of f is obtained by attaching the weight f/g to each sample $\mathbf{x}$

Figure source: Thrun, Burgard, Fox



Importance Sampling for Bayesian Estimation

$$\begin{aligned}\mathbb{E}[f(X)] &= \int_X f(\mathbf{x}_{0:t})p(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})d\mathbf{x}_{0:t} \\ &= \int_X f(\mathbf{x}_{0:t})\frac{p(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})}{q(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})}q(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})d\mathbf{x}_{0:t}\end{aligned}$$

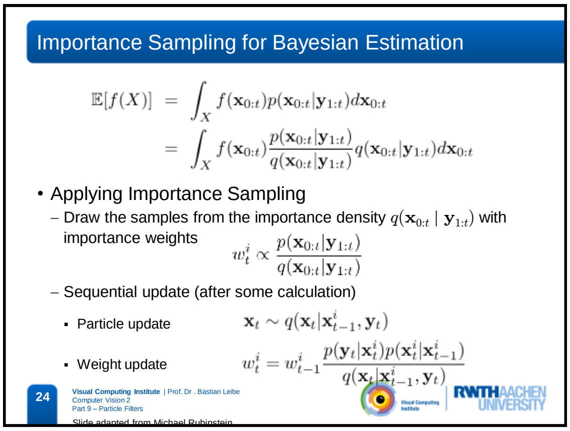
- Applying Importance Sampling
  - Characterize the posterior pdf using a set of samples (particles) and their weights

$$\{\mathbf{x}^i_{0:t}, w^i_t\}^N_{i=1}$$

  - Then the joint posterior is approximated by

$$p(\mathbf{x}_{0:t}|\mathbf{y}_{1:t}) \approx \sum_{i=1}^N w^i_t \delta(\mathbf{x}_{0:t} - \mathbf{x}^i_{0:t})$$

Slide adapted from Michael Rubinstein

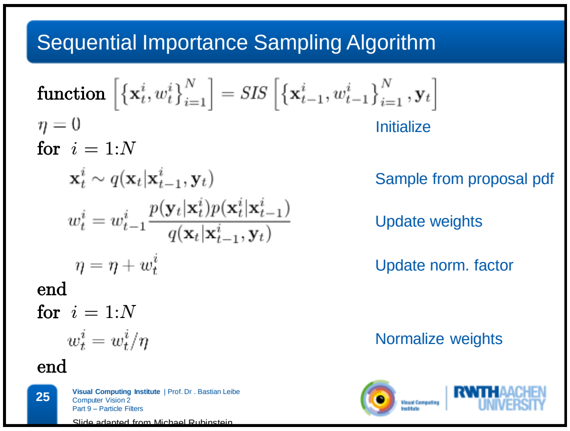


Importance Sampling for Bayesian Estimation

$$\begin{aligned}\mathbb{E}[f(X)] &= \int_X f(\mathbf{x}_{0:t})p(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})d\mathbf{x}_{0:t} \\ &= \int_X f(\mathbf{x}_{0:t})\frac{p(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})}{q(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})}q(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})d\mathbf{x}_{0:t}\end{aligned}$$

- Applying Importance Sampling
  - Draw the samples from the importance density $q(\mathbf{x}_{0:t} \mid \mathbf{y}_{1:t})$ with importance weights $w^i_t \propto \frac{p(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})}{q(\mathbf{x}_{0:t}|\mathbf{y}_{1:t})}$
  - Sequential update (after some calculation)
    - Particle update $\mathbf{x}_t \sim q(\mathbf{x}_t|\mathbf{x}^i_{t-1}, \mathbf{y}_t)$
    - Weight update $w^i_t = w^i_{t-1}\frac{p(\mathbf{y}_t|\mathbf{x}^i_t)p(\mathbf{x}^i_t|\mathbf{x}^i_{t-1})}{q(\mathbf{x}_t|\mathbf{x}^i_{t-1}, \mathbf{y}_t)}$

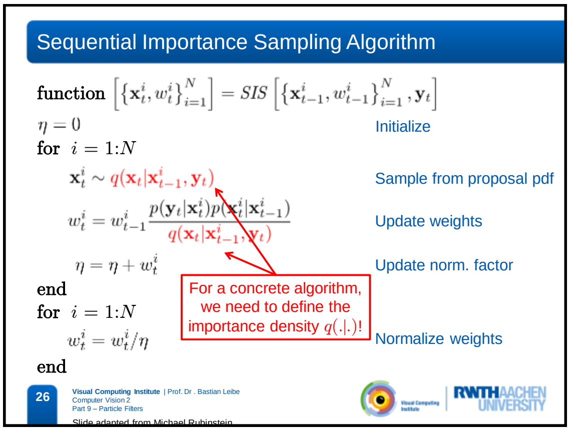
Slide adapted from Michael Rubinstein



Sequential Importance Sampling Algorithm

```
function $\left[\{\mathbf{x}^i_t, w^i_t\}^N_{i=1}\right] = SIS\left[\{\mathbf{x}^i_{t-1}, w^i_{t-1}\}^N_{i=1}, \mathbf{y}_t\right]$
$\eta = 0$                                                        Initialize
for $i = 1{:}N$
    $\mathbf{x}^i_t \sim q(\mathbf{x}_t|\mathbf{x}^i_{t-1}, \mathbf{y}_t)$                          Sample from proposal pdf
    $w^i_t = w^i_{t-1}\frac{p(\mathbf{y}_t|\mathbf{x}^i_t)p(\mathbf{x}^i_t|\mathbf{x}^i_{t-1})}{q(\mathbf{x}_t|\mathbf{x}^i_{t-1}, \mathbf{y}_t)}$     Update weights
    $\eta = \eta + w^i_t$                                        Update norm. factor
end
for $i = 1{:}N$
    $w^i_t = w^i_t/\eta$                                         Normalize weights
end
```

Slide adapted from Michael Rubinstein

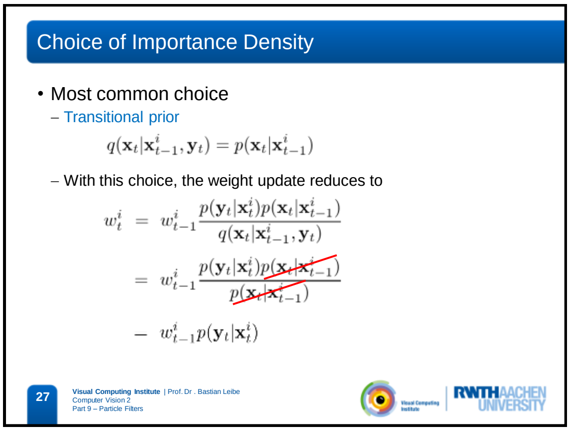


Sequential Importance Sampling Algorithm

```
function $\left[\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N\right] = SIS\left[\{\mathbf{x}_{t-1}^i, w_{t-1}^i\}_{i=1}^N, \mathbf{y}_t\right]$
$\eta = 0$                                                      Initialize
for  $i = 1{:}N$
    $\mathbf{x}_t^i \sim q(\mathbf{x}_t|\mathbf{x}_{t-1}^i, \mathbf{y}_t)$                      Sample from proposal pdf
    $w_t^i = w_{t-1}^i \frac{p(\mathbf{y}_t|\mathbf{x}_t^i)p(\mathbf{x}_t^i|\mathbf{x}_{t-1}^i)}{q(\mathbf{x}_t|\mathbf{x}_{t-1}^i, \mathbf{y}_t)}$        Update weights
    $\eta = \eta + w_t^i$                                       Update norm. factor
end
for  $i = 1{:}N$
    $w_t^i = w_t^i/\eta$                                        Normalize weights
end
```

For a concrete algorithm, we need to define the importance density $q(.|.)$!

Slide adapted from Michael Rubinstein



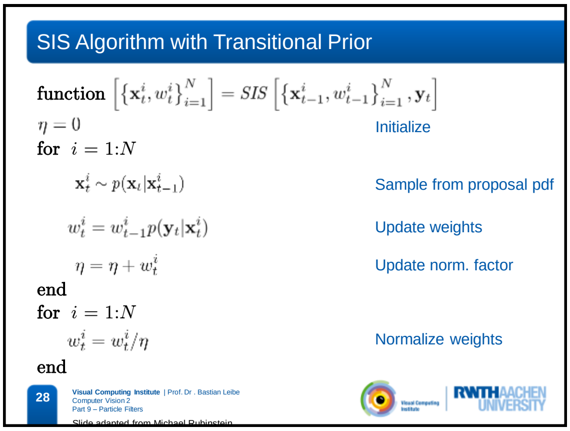
Choice of Importance Density

- Most common choice
  - Transitional prior

$$q(\mathbf{x}_t|\mathbf{x}_{t-1}^i, \mathbf{y}_t) = p(\mathbf{x}_t|\mathbf{x}_{t-1}^i)$$

  - With this choice, the weight update reduces to

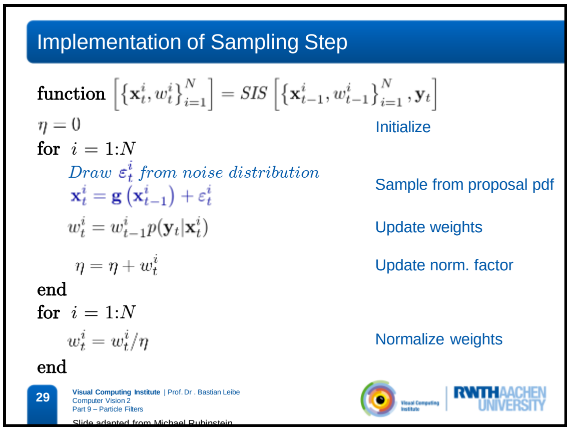
$$\begin{aligned} w_t^i &= w_{t-1}^i \frac{p(\mathbf{y}_t|\mathbf{x}_t^i)p(\mathbf{x}_t|\mathbf{x}_{t-1}^i)}{q(\mathbf{x}_t|\mathbf{x}_{t-1}^i, \mathbf{y}_t)} \\ &= w_{t-1}^i \frac{p(\mathbf{y}_t|\mathbf{x}_t^i)\cancel{p(\mathbf{x}_t|\mathbf{x}_{t-1}^i)}}{\cancel{p(\mathbf{x}_t|\mathbf{x}_{t-1}^i)}} \\ &= w_{t-1}^i p(\mathbf{y}_t|\mathbf{x}_t^i) \end{aligned}$$



SIS Algorithm with Transitional Prior

```
function $\left[\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N\right] = SIS\left[\{\mathbf{x}_{t-1}^i, w_{t-1}^i\}_{i=1}^N, \mathbf{y}_t\right]$
$\eta = 0$                                                      Initialize
for  $i = 1{:}N$
    $\mathbf{x}_t^i \sim p(\mathbf{x}_t|\mathbf{x}_{t-1}^i)$                                 Sample from proposal pdf
    $w_t^i = w_{t-1}^i p(\mathbf{y}_t|\mathbf{x}_t^i)$                               Update weights
    $\eta = \eta + w_t^i$                                       Update norm. factor
end
for  $i = 1{:}N$
    $w_t^i = w_t^i/\eta$                                        Normalize weights
end
```

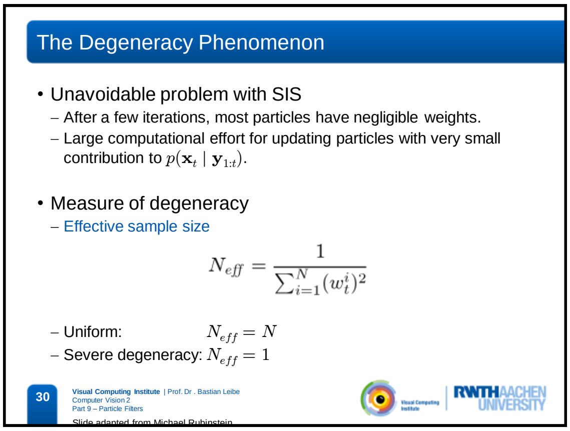
Slide adapted from Michael Rubinstein



Implementation of Sampling Step

```
function $\left[\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N\right] = SIS\left[\{\mathbf{x}_{t-1}^i, w_{t-1}^i\}_{i=1}^N, \mathbf{y}_t\right]$
$\eta = 0$                                                      Initialize
for  $i = 1{:}N$
    $\textit{Draw } \varepsilon_t^i \textit{ from noise distribution}$
    $\mathbf{x}_t^i = \mathbf{g}\left(\mathbf{x}_{t-1}^i\right) + \varepsilon_t^i$                          Sample from proposal pdf
    $w_t^i = w_{t-1}^i p(\mathbf{y}_t|\mathbf{x}_t^i)$                               Update weights
    $\eta = \eta + w_t^i$                                       Update norm. factor
end
for  $i = 1{:}N$
    $w_t^i = w_t^i/\eta$                                        Normalize weights
end
```

Slide adapted from Michael Rubinstein



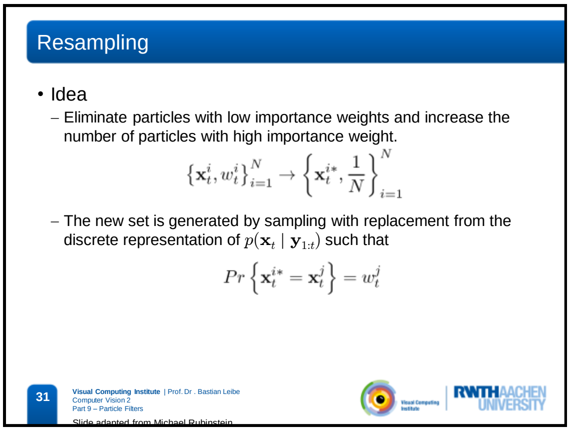
The Degeneracy Phenomenon

- Unavoidable problem with SIS
  - After a few iterations, most particles have negligible weights.
  - Large computational effort for updating particles with very small contribution to $p(\mathbf{x}_t \mid \mathbf{y}_{1:t})$.
- Measure of degeneracy
  - Effective sample size

$$N_{eff} = \frac{1}{\sum_{i=1}^N (w_t^i)^2}$$

  - Uniform: $N_{eff} = N$
  - Severe degeneracy: $N_{eff} = 1$

Slide adapted from Michael Rubinstein



Resampling

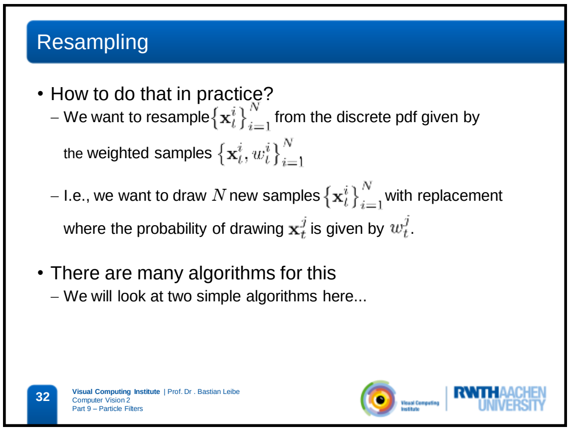
- Idea
  - Eliminate particles with low importance weights and increase the number of particles with high importance weight.

$$\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N \to \left\{\mathbf{x}_t^{i*}, \frac{1}{N}\right\}_{i=1}^N$$

  - The new set is generated by sampling with replacement from the discrete representation of $p(\mathbf{x}_t \mid \mathbf{y}_{1:t})$ such that

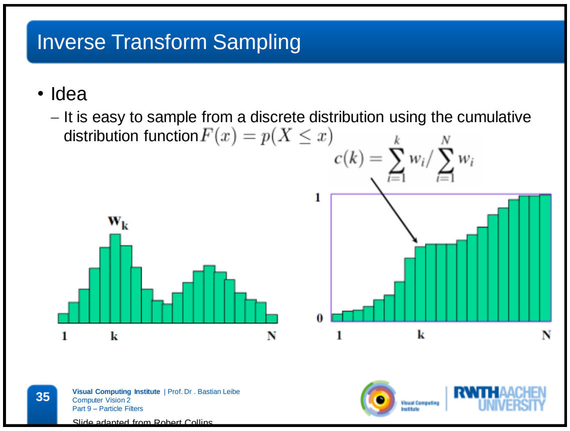
$$Pr\left\{\mathbf{x}_t^{i*} = \mathbf{x}_t^j\right\} = w_t^j$$

Slide adapted from Michael Rubinstein



Resampling

- How to do that in practice?
  - We want to resample $\{\mathbf{x}_t^i\}_{i=1}^N$ from the discrete pdf given by the weighted samples $\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N$
  - I.e., we want to draw N new samples $\{\mathbf{x}_t^i\}_{i=1}^N$ with replacement where the probability of drawing $\mathbf{x}_t^j$ is given by w_t^j.
- There are many algorithms for this
  - We will look at two simple algorithms here...

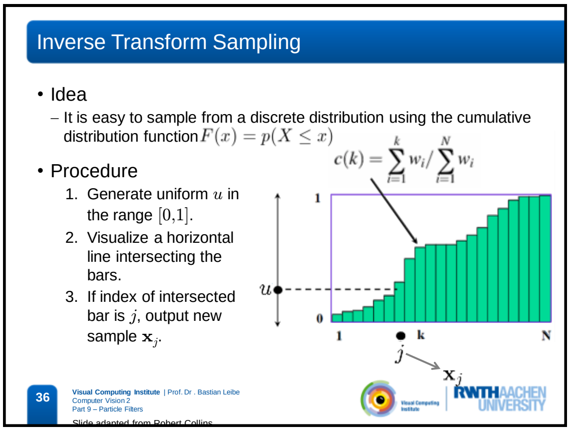


Inverse Transform Sampling

- Idea
  - It is easy to sample from a discrete distribution using the cumulative distribution function $F(x) = p(X \leq x)$

$$c(k) = \sum_{i=1}^{k} w_i / \sum_{i=1}^{N} w_i$$

Slide adapted from Robert Collins

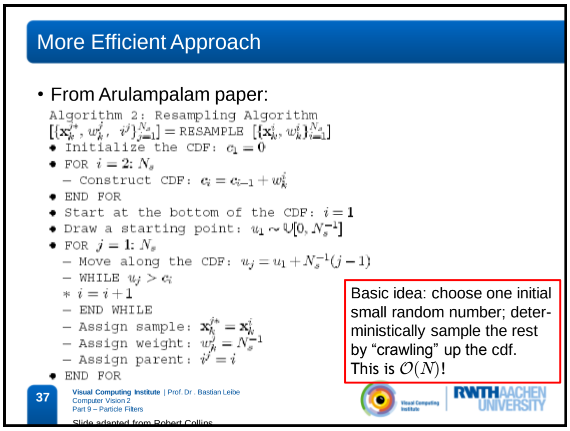


Inverse Transform Sampling

- Idea
  - It is easy to sample from a discrete distribution using the cumulative distribution function $F(x) = p(X \leq x)$
- Procedure
  1. Generate uniform u in the range $[0,1]$.
  2. Visualize a horizontal line intersecting the bars.
  3. If index of intersected bar is j, output new sample $\mathbf{x}_j$.

$$c(k) = \sum_{i=1}^{k} w_i / \sum_{i=1}^{N} w_i$$

Slide adapted from Robert Collins



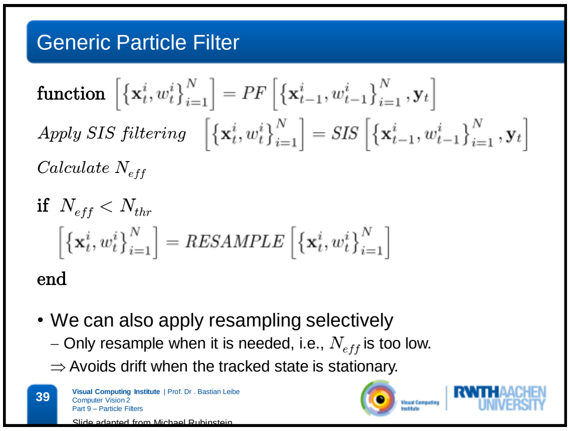
More Efficient Approach

- From Arulampalam paper:

```
Algorithm 2: Resampling Algorithm
$[\{\mathbf{x}_k^{j*}, w_k^j, i^j\}_{j=1}^{N_s}] = $ RESAMPLE $[\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}]$
• Initialize the CDF: $c_1 = 0$
• FOR $i = 2: N_s$
  – Construct CDF: $c_i = c_{i-1} + w_k^i$
• END FOR
• Start at the bottom of the CDF: $i = 1$
• Draw a starting point: $u_1 \sim \mathbb{U}[0, N_s^{-1}]$
• FOR $j = 1: N_s$
  – Move along the CDF: $u_j = u_1 + N_s^{-1}(j - 1)$
  – WHILE $u_j > c_i$
  * $i = i + 1$
  – END WHILE
  – Assign sample: $\mathbf{x}_k^{j*} = \mathbf{x}_k^i$
  – Assign weight: $w_k^j = N_s^{-1}$
  – Assign parent: $i^j = i$
• END FOR
```

Basic idea: choose one initial small random number; deterministically sample the rest by “crawling” up the cdf. This is $\mathcal{O}(N)$!

Slide adapted from Robert Collins



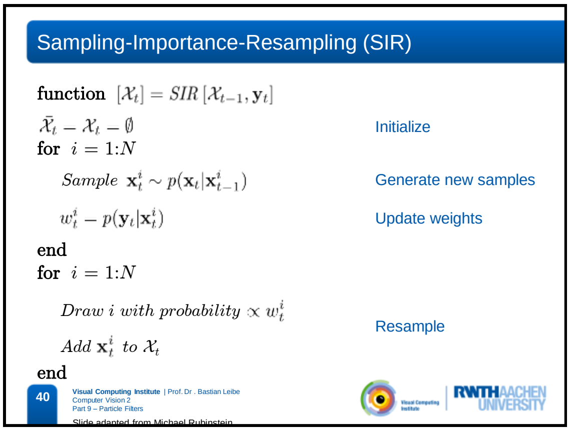
Generic Particle Filter

```
function $\left[\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N\right] = PF\left[\{\mathbf{x}_{t-1}^i, w_{t-1}^i\}_{i=1}^N, \mathbf{y}_t\right]$
Apply SIS filtering $\left[\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N\right] = SIS\left[\{\mathbf{x}_{t-1}^i, w_{t-1}^i\}_{i=1}^N, \mathbf{y}_t\right]$
Calculate $N_{eff}$
if $N_{eff} < N_{thr}$
    $\left[\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N\right] = RESAMPLE\left[\{\mathbf{x}_t^i, w_t^i\}_{i=1}^N\right]$
end
```

- We can also apply resampling selectively
  - Only resample when it is needed, i.e., N_{eff} is too low.

  ⇒ Avoids drift when the tracked state is stationary.

Slide adapted from Michael Rubinstein



Sampling-Importance-Resampling (SIR)

```
function $[\mathcal{X}_t] = SIR\,[\mathcal{X}_{t-1}, \mathbf{y}_t]$
$\bar{\mathcal{X}}_t - \mathcal{X}_t - \emptyset$                      Initialize
for $i = 1{:}N$
    Sample $\mathbf{x}_t^i \sim p(\mathbf{x}_t|\mathbf{x}_{t-1}^i)$      Generate new samples
    $w_t^i - p(\mathbf{y}_t|\mathbf{x}_t^i)$                   Update weights
end
for $i = 1{:}N$
    Draw $i$ with probability $\propto w_t^i$                       Resample
    Add $\mathbf{x}_t^i$ to $\mathcal{X}_t$
end
```

Slide adapted from Michael Rubinstein

Other Variant of the Algorithm

```
function  $[\mathcal{X}_t] = SIR\,[\mathcal{X}_{t-1}, \mathbf{y}_t]$
$\bar{\mathcal{X}}_t - \mathcal{X}_t - \emptyset$
for  $i = 1{:}N$
    $Sample\ \mathbf{x}_t^i \sim p(\mathbf{x}_t|\mathbf{x}_{t-1}^i)$
    $w_t^i - p(\mathbf{y}_t|\mathbf{x}_t^i)$
end
for  $i = 1{:}N$
    $Draw\ i\ with\ probability$
    $Add\ \mathbf{x}_t^i\ to\ \mathcal{X}_t$
end
```

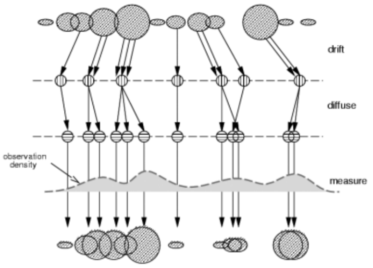
Important property:

Particles are distributed according to pdf from previous time step.

Particles are distributed according to posterior from this time step.

Slide adapted from Michael Rubinstein

Recap: Condensation Algorithm



Start with weighted samples from previous time step

Sample and shift according to dynamics model

Spread due to randomness; this is predicted density $P(X_t|Y_{t-1})$

Weight the samples according to observation density

Arrive at corrected density estimate $P(X_t|Y_t)$

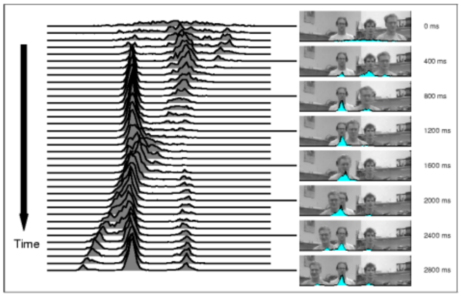
M. Isard and A. Blake, CONDENSATION -- conditional density propagation for visual tracking, IJCV 29(1):5-28, 1998

Slide credit: Svetlana Lazebnik

Particle Filtering – Visualization

Code and video available from
http://www.robots.ox.ac.uk/~misard/condensation.html

Particle Filtering Results



http://www.robots.ox.ac.uk/~misard/condensation.html

Particle Filtering Results

- Some more examples

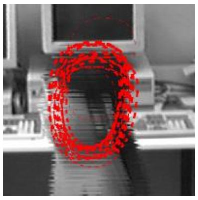
http://www.robots.ox.ac.uk/~misard/condensation.html

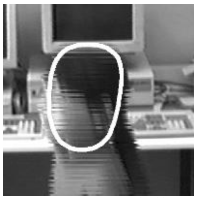
Videos from Isard & Blake

Sidenote: Obtaining a State Estimate

- Note that there’s no explicit state estimate maintained, just a “cloud” of particles
- Can obtain an estimate at a particular time by querying the current particle set
- Some approaches
  - “Mean” particle
    - Weighted sum of particles
    - Confidence: inverse variance
  - Really want a mode finder—mean of tallest peak

Condensation: Estimating Target State





From Isard & Blake, 1998

State samples
(thickness proportional to weight)

Mean of weighted
state samples

Slide credit: Marc Pollefeys

Figures from Isard & Blake

Summary: Particle Filtering

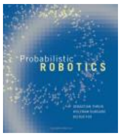
- Pros:
  - Able to represent arbitrary densities
  - Converging to true posterior even for non-Gaussian and nonlinear system
  - Efficient: particles tend to focus on regions with high probability
  - Works with many different state spaces
    - E.g. articulated tracking in complicated joint angle spaces
  - Many extensions available

Summary: Particle Filtering

- Cons / Caveats:
  - #Particles is important performance factor
    - Want as few particles as possible for efficiency.
    - But need to cover state space sufficiently well.
  - Worst-case complexity grows exponentially in the dimensions
  - Multimodal densities possible, but still single object
    - Interactions between multiple objects require special treatment.
    - Not handled well in the particle filtering framework (state space explosion).

References and Further Reading

- A good description of Particle Filters can be found in Ch.4.3 of the following book
  - S. Thrun, W. Burgard, D. Fox. Probabilistic Robotics. MIT Press, 2006.



- A good tutorial on Particle Filters
  - M.S. Arulampalam, S. Maskell, N. Gordon, T. Clapp. A Tutorial on Particle Filters for Online Nonlinear/Non-Gaussian Bayesian Tracking. In *IEEE Transactions on Signal Processing*, Vol. 50(2), pp. 174-188, 2002.
- The CONDENSATION paper
  - M. Isard and A. Blake, CONDENSATION - conditional density propagation for visual tracking, IJCV 29(1):5-28, 1998