Machine Learning – Lecture 8

Nonlinear Support Vector Machines

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Announcements

• Exam dates
  - 1st date: Monday, 07.03., 13:30h – 16:00h
  - 2nd date: Monday, 29.03., 10:30h – 13:00h

  - The lecture dates have been optimized to avoid overlaps with other Computer Science Master lectures as much as possible.
  - If you still have conflicts with both exam dates, please tell us.
  - If you’re not a CS/SSE/MI student and want to take the exam and cannot register on Campus, please do NOT yet register with us.
    - We will collect those registrations in mid-January

• Please register for the exam on Campus until next week Friday (17.11.)!
Course Outline

• Fundamentals
  ➢ Bayes Decision Theory
  ➢ Probability Density Estimation

• Classification Approaches
  ➢ Linear Discriminants
  ➢ Support Vector Machines
  ➢ Ensemble Methods & Boosting
  ➢ Randomized Trees, Forests & Ferns

• Deep Learning
  ➢ Foundations
  ➢ Convolutional Neural Networks
  ➢ Recurrent Neural Networks
Topics of This Lecture

• Support Vector Machines
  - Recap: Lagrangian (primal) formulation
  - Dual formulation
  - Soft-margin classification

• Nonlinear Support Vector Machines
  - Nonlinear basis functions
  - The Kernel trick
  - Mercer’s condition
  - Popular kernels

• Analysis
  - Error function

• Applications
Recap: Support Vector Machine (SVM)

• Basic idea
  - The SVM tries to find a classifier which maximizes the margin between pos. and neg. data points.
  - Up to now: consider linear classifiers
    \[ \mathbf{w}^T \mathbf{x} + b = 0 \]

• Formulation as a convex optimization problem
  - Find the hyperplane satisfying
    \[ \arg \min_{\mathbf{w},b} \frac{1}{2} \| \mathbf{w} \|^2 \]
    under the constraints
    \[ t_n (\mathbf{w}^T \mathbf{x}_n + b) \geq 1 \quad \forall n \]
    based on training data points \( \mathbf{x}_n \) and target values \( t_n \in \{-1, 1\} \)
Recap: SVM – Lagrangian Formulation

- Find hyperplane minimizing $\|w\|^2$ under the constraints
  \[ t \sum \left( w^T x_n + b \right) - 1 \geq 0 \quad \forall n \]

- Lagrangian formulation
  - Introduce positive Lagrange multipliers: $a_n \geq 0 \quad \forall n$
  - Minimize Lagrangian ("primal form")
    \[ L(w, b, a) = \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n \left( t_n (w^T x_n + b) - 1 \right) \]
  - I.e., find $w$, $b$, and $a$ such that
    \[ \frac{\partial L}{\partial b} = 0 \Rightarrow \sum_{n=1}^{N} a_n t_n = 0 \quad \frac{\partial L}{\partial w} = 0 \Rightarrow w = \sum_{n=1}^{N} a_n t_n x_n \]
Recap: SVM – Primal Formulation

- Lagrangian primal form

\[
L_p = \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n \left\{ t_n (w^T x_n + b) - 1 \right\}
\]

\[
= \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n \left\{ t_n y(x_n) - 1 \right\}
\]

- The solution of \( L_p \) needs to fulfill the KKT conditions
  - Necessary and sufficient conditions

  \[
  a_n \geq 0
  \]

  \[
  t_n y(x_n) - 1 \geq 0
  \]

  \[
  a_n \left\{ t_n y(x_n) - 1 \right\} = 0
  \]

  \[
  \lambda \geq 0
  \]

  \[
  f(x) \geq 0
  \]

  \[
  \lambda f(x) = 0
  \]
SVM – Solution (Part 1)

- Solution for the hyperplane
  - Computed as a linear combination of the training examples
    \[
    \mathbf{w} = \sum_{n=1}^{N} a_n t_n \mathbf{x}_n
    \]
  - Because of the KKT conditions, the following must also hold
    \[
    a_n \left(t_n (\mathbf{w}^T \mathbf{x}_n + b) - 1\right) = 0
    \]
  - This implies that \(a_n > 0\) only for training data points for which
    \[
    \left(t_n (\mathbf{w}^T \mathbf{x}_n + b) - 1\right) = 0
    \]

⇒ Only some of the data points actually influence the decision boundary!

Slide adapted from Bernt Schiele
SVM – Support Vectors

• The training points for which \( a_n > 0 \) are called “support vectors”.

• Graphical interpretation:
  - The support vectors are the points on the margin.
  - They *define* the margin and thus the hyperplane.

⇒ Robustness to “too correct” points!

Image source: C. Burges, 1998
SVM – Solution (Part 2)

- **Solution for the hyperplane**
  - To define the decision boundary, we still need to know $b$.
  - Observation: any support vector $x_n$ satisfies
    
    $t_n y(x_n) = t_n \left( \sum_{m \in S} a_m t_m x_m^T x_n + b \right) = 1$

  - Using $t_n^2 = 1$ we can derive: $b = t_n - \sum_{m \in S} a_m t_m x_m^T x_n$

  - In practice, it is more robust to average over all support vectors:
    
    $b = \frac{1}{N_S} \sum_{n \in S} \left( t_n - \sum_{m \in S} a_m t_m x_m^T x_n \right)$
SVM – Discussion (Part 1)

• Linear SVM
  ➢ Linear classifier
  ➢ SVMs have a “guaranteed” generalization capability.
  ➢ Formulation as convex optimization problem.
    ⇒ Globally optimal solution!

• Primal form formulation
  ➢ Solution to quadratic prog. problem in $M$ variables is in $O(M^3)$.
  ➢ Here: $D$ variables ⇒ $O(D^3)$
  ➢ Problem: scaling with high-dim. data (“curse of dimensionality”)
SVM – Dual Formulation

• Improving the scaling behavior: rewrite $L_p$ in a dual form

$$L_p = \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n \left\{ t_n (w^T x_n + b) - 1 \right\}$$

$$= \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n t_n w^T x_n - b \sum_{n=1}^{N} a_n t_n + \sum_{n=1}^{N} a_n$$

- Using the constraint $\sum_{n=1}^{N} a_n t_n = 0$ we obtain

$$\frac{\partial L_p}{\partial b} = 0$$

$$L_p = \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n t_n w^T x_n + \sum_{n=1}^{N} a_n$$
SVM – Dual Formulation

\[ L_p = \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n t_n w^T x_n + \sum_{n=1}^{N} a_n \]

- Using the constraint \( w = \sum_{n=1}^{N} a_n t_n x_n \) we obtain

\[ \frac{\partial L_p}{\partial w} = 0 \]

\[ L_p = \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} a_n t_n \sum_{m=1}^{N} a_m t_m x_m^T x_n + \sum_{n=1}^{N} a_n \]

\[ = \frac{1}{2} \|w\|^2 - \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m (x_m^T x_n) + \sum_{n=1}^{N} a_n \]

Slide adapted from Bernt Schiele
SVM – Dual Formulation

\[ L = \frac{1}{2} \| \mathbf{w} \|^2 - \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m (\mathbf{x}_m^T \mathbf{x}_n) + \sum_{n=1}^{N} a_n \]

- Applying \( \frac{1}{2} \| \mathbf{w} \|^2 = \frac{1}{2} \mathbf{w}^T \mathbf{w} \) and again using \( \mathbf{w} = \sum_{n=1}^{N} a_n t_n \mathbf{x}_n \)

\[ \frac{1}{2} \mathbf{w}^T \mathbf{w} = \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m (\mathbf{x}_m^T \mathbf{x}_n) \]

- Inserting this, we get the Wolfe dual

\[ L_d(\mathbf{a}) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m (\mathbf{x}_m^T \mathbf{x}_n) \]

Slide adapted from Bernt Schiele
SVM – Dual Formulation

• Maximize

\[ L_d(a) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m (x_m^T x_n) \]

under the conditions

\[ a_n \geq 0 \quad \forall n \]

\[ \sum_{n=1}^{N} a_n t_n = 0 \]

➢ The hyperplane is given by the \( N_S \) support vectors:

\[ w = \sum_{n=1}^{N_S} a_n t_n x_n \]
SVM – Discussion (Part 2)

• Dual form formulation
  - In going to the dual, we now have a problem in $N$ variables ($a_n$).
  - Isn’t this worse??? We penalize large training sets!

• However…
  1. SVMs have sparse solutions: $a_n \neq 0$ only for support vectors!
     ⇒ This makes it possible to construct efficient algorithms
        – e.g. Sequential Minimal Optimization (SMO)
        – Effective runtime between $O(N)$ and $O(N^2)$.
  2. We have avoided the dependency on the dimensionality.
     ⇒ This makes it possible to work with infinite-dimensional feature
        spaces by using suitable basis functions $\phi(x)$.
     ⇒ We’ll see that later in today’s lecture…
So Far…

• Only looked at linearly separable case…
  - Current problem formulation has no solution if the data are not linearly separable!
  - Need to introduce some tolerance to outlier data points.
SVM – Non-Separable Data

• Non-separable data
  
  i.e. the following inequalities cannot be satisfied for all data points

\[
\begin{align*}
\mathbf{w}^T \mathbf{x}_n + b & \geq +1 \quad \text{for} \quad t_n = +1 \\
\mathbf{w}^T \mathbf{x}_n + b & \leq -1 \quad \text{for} \quad t_n = -1
\end{align*}
\]

• Instead use

\[
\begin{align*}
\mathbf{w}^T \mathbf{x}_n + b & \geq +1 - \xi_n \quad \text{for} \quad t_n = +1 \\
\mathbf{w}^T \mathbf{x}_n + b & \leq -1 + \xi_n \quad \text{for} \quad t_n = -1
\end{align*}
\]

with “slack variables”

\[
\xi_n \geq 0 \quad \forall n
\]
SVM – Soft-Margin Classification

- Slack variables
  - One slack variable $\xi_n \geq 0$ for each training data point.

- Interpretation
  - $\xi_n = 0$ for points that are on the correct side of the margin.
  - $\xi_n = |t_n - y(x_n)|$ for all other points (linear penalty).
  - We do not have to set the slack variables ourselves!
    ⇒ They are jointly optimized together with $w$. 

$w$ Point on decision boundary: $\xi_n = 1$

Misclassified point: $\xi_n > 1$

How that?

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SVM – Non-Separable Data

- Separable data
  - Minimize

- Non-separable data
  - Minimize

\[
\frac{1}{2} \left\| w \right\|^2
\]

Trade-off parameter!
SVM – New Primal Formulation

• New SVM Primal: Optimize

\[ L_p = \frac{1}{2} \|w\|^2 + C \sum_{n=1}^{N} \xi_n - \sum_{n=1}^{N} a_n (t_n y(x_n) - 1 + \xi_n) - \sum_{n=1}^{N} \mu_n \xi_n \]

Constraint
\[ t_n y(x_n) \geq 1 - \xi_n \]
Constraint
\[ \xi_n \geq 0 \]

• KKT conditions

\[ a_n \geq 0 \quad \mu_n \geq 0 \]
\[ t_n y(x_n) - 1 + \xi_n \geq 0 \quad \xi_n \geq 0 \]
\[ a_n (t_n y(x_n) - 1 + \xi_n) = 0 \quad \mu_n \xi_n = 0 \]

KKT:
\[ \lambda \geq 0 \]
\[ f(x) \geq 0 \]
\[ \lambda f(x) = 0 \]
SVM – New Dual Formulation

- New SVM Dual: Maximize

\[ L_d(a) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_na_m t_n t_m (x_m^T x_n) \]

under the conditions

\[ 0 \cdot a_n \cdot C \]
\[ \sum_{n=1}^{N} a_n t_n = 0 \]

- This is again a quadratic programming problem
  \[ \Rightarrow \] Solve as before… (more on that later)

This is all that changed!
SVM – New Solution

• Solution for the hyperplane
  ➢ Computed as a linear combination of the training examples

\[
\mathbf{w} = \sum_{n=1}^{N} a_n t_n \mathbf{x}_n
\]

➢ Again sparse solution: \(a_n = 0\) for points outside the margin.
⇒ The slack points with \(\xi_n > 0\) are now also support vectors!

➢ Compute \(b\) by averaging over all \(N_M\) points with \(0 < a_n < C\):

\[
b = \frac{1}{N_M} \sum_{n \in M} \left( t_n - \sum_{m \in M} a_m t_m x_m^T x_n \right)
\]
Interpretation of Support Vectors

- Those are the hard examples!
  - We can visualize them, e.g. for face detection

Image source: E. Osuna, F. Girosi, 1997
Topics of This Lecture

• Support Vector Machines
  - Recap: Lagrangian (primal) formulation
  - Dual formulation
  - Soft-margin classification

• Nonlinear Support Vector Machines
  - Nonlinear basis functions
  - The Kernel trick
  - Mercer’s condition
  - Popular kernels

• Analysis
  - Error function

• Applications
So Far…

• Only looked at linearly separable case…
  ➢ Current problem formulation has no solution if the data are not linearly separable!
  ➢ Need to introduce some tolerance to outlier data points.
  ⇒ Slack variables.

• Only looked at linear decision boundaries…
  ➢ This is not sufficient for many applications.
  ➢ Want to generalize the ideas to non-linear boundaries.
Nonlinear SVM

• Linear SVMs
  - Datasets that are linearly separable with some noise work well:
    - But what are we going to do if the dataset is just too hard?
  - How about... mapping data to a higher-dimensional space:
Nonlinear SVM – Feature Spaces

• General idea: The original input space can be mapped to some higher-dimensional feature space where the training set is separable:

\[ \Phi: x \rightarrow \phi(x) \]
Nonlinear SVM

• General idea
  
  - Nonlinear transformation $\phi$ of the data points $x_n$:

  $$x \in \mathbb{R}^D \quad \phi : \mathbb{R}^D \rightarrow \mathcal{H}$$

  - Hyperplane in higher-dim. space $\mathcal{H}$ (linear classifier in $\mathcal{H}$)

  $$\mathbf{w}^T \phi(x) + b = 0$$

  $\Rightarrow$ Nonlinear classifier in $\mathbb{R}^D$. 

Slide credit: Bernt Schiele
What Could This Look Like?

• Example:
  - Mapping to polynomial space, \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \):

\[
\phi(\mathbf{x}) = \begin{bmatrix}
x_1^2 \\
\sqrt{2}x_1x_2 \\
x_2^2
\end{bmatrix}
\]

  - Motivation: Easier to separate data in higher-dimensional space.
  - But wait – isn’t there a big problem?
    - How should we evaluate the decision function?

Image source: C. Burges, 1998
Problem with High-dim. Basis Functions

• Problem
  ➢ In order to apply the SVM, we need to evaluate the function
    \[ y(x) = w^T \phi(x) + b \]
  ➢ Using the hyperplane, which is itself defined as
    \[ w = \sum_{n=1}^{N} a_n t_n \phi(x_n) \]

⇒ What happens if we try this for a million-dimensional feature space \( \phi(x) \)?
  ➢ Oh-oh…
Solution: The Kernel Trick

• Important observation
  - \( \phi(x) \) only appears in the form of dot products \( \phi(x)^T\phi(y) \):

\[
y(x) = w^T\phi(x) + b
= \sum_{n=1}^{N} a_n t_n \phi(x_n)^T\phi(x) + b
\]

  - Trick: Define a so-called kernel function \( k(x,y) = \phi(x)^T\phi(y) \).
  - Now, in place of the dot product, use the kernel instead:

\[
y(x) = \sum_{n=1}^{N} a_n t_n k(x_n, x) + b
\]

  - The kernel function *implicitly* maps the data to the higher-dimensional space (without having to compute \( \phi(x) \) explicitly)!
Back to Our Previous Example…

• 2\textsuperscript{nd} degree polynomial kernel:

\[
\phi(x)^T \phi(y) = \begin{bmatrix} x_1^2 \\ \sqrt{2}x_1 x_2 \\ x_2^2 \end{bmatrix} \cdot \begin{bmatrix} y_1^2 \\ \sqrt{2}y_1 y_2 \\ y_2^2 \end{bmatrix} \\
= x_1^2 y_1^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_2^2 \\
= (x^T y)^2 =: k(x, y)
\]

Whenever we evaluate the kernel function \( k(x, y) = (x^T y)^2 \), we implicitly compute the dot product in the higher-dimensional feature space.
SVMs with Kernels

• Using kernels
  - Applying the kernel trick is easy. Just replace every dot product by a kernel function…
    \[ x^T y \rightarrow k(x, y) \]
  - …and we’re done.
  - Instead of the raw input space, we’re now working in a higher-dimensional (potentially infinite dimensional!) space, where the data is more easily separable.

“Sounds like magic…”

• Wait – does this always work?
  - The kernel needs to define an implicit mapping to a higher-dimensional feature space \( \phi(x) \).
  - When is this the case?
Which Functions are Valid Kernels?

- Mercer’s theorem (modernized version):
  
  Every positive definite symmetric function is a kernel.

- Positive definite symmetric functions correspond to a positive definite symmetric Gram matrix:

\[
\begin{bmatrix}
  k(x_1, x_1) & k(x_1, x_2) & k(x_1, x_3) & \cdots & k(x_1, x_n) \\
  k(x_2, x_1) & k(x_2, x_2) & k(x_2, x_3) & \cdots & k(x_2, x_n) \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  k(x_n, x_1) & k(x_n, x_2) & k(x_n, x_3) & \cdots & k(x_n, x_n)
\end{bmatrix}
\]

(positive definite = all eigenvalues are > 0)

Slide credit: Raymond Mooney
Kernels Fulfilling Mercer’s Condition

- Polynomial kernel
  \[ k(x, y) = (x^T y + 1)^p \]

- Radial Basis Function kernel
  \[ k(x, y) = \exp \left\{ -\frac{(x - y)^2}{2\sigma^2} \right\} \]
  e.g. Gaussian

- Hyperbolic tangent kernel
  \[ k(x, y) = \tanh(\lambda x^T y + \delta) \]
  e.g. Sigmoid
  (and many, many more…)

Actually, this was wrong in the original SVM paper...

Slide credit: Bernt Schiele

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Example: Bag of Visual Words Representation

- General framework in visual recognition
  - Create a codebook (vocabulary) of prototypical image features
  - Represent images as histograms over codebook activations
  - Compare two images by any histogram kernel, e.g. $\chi^2$ kernel

\[
k_{\chi^2}(h, h') = \exp \left( -\frac{1}{\gamma} \sum_j \frac{(h_j - h'_j)^2}{h_j + h'_j} \right)
\]
Nonlinear SVM – Dual Formulation

- SVM Dual: Maximize

\[ L_d(a) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m k(x_m, x_n) \]

under the conditions

\[ 0 \cdot a_n \cdot C \]
\[ \sum_{n=1}^{N} a_n t_n = 0 \]

- Classify new data points using

\[ y(x) = \sum_{n=1}^{N} a_n t_n k(x_n, x) + b \]
SVM Demo

Applet from libsvm
(http://www.csie.ntu.edu.tw/~cjlin/libsvm/)

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Summary: SVMs

• Properties
  ➢ Empirically, SVMs work very, very well.
  ➢ SVMs are currently among the best performers for a number of classification tasks ranging from text to genomic data.
  ➢ SVMs can be applied to complex data types beyond feature vectors (e.g. graphs, sequences, relational data) by designing kernel functions for such data.
  ➢ SVM techniques have been applied to a variety of other tasks – e.g. SV Regression, One-class SVMs, …
  ➢ The kernel trick has been used for a wide variety of applications. It can be applied wherever dot products are in use – e.g. Kernel PCA, kernel FLD, …
  ➢ Good overview, software, and tutorials available on http://www.kernel-machines.org/
Summary: SVMs

- Limitations
  - How to select the right kernel?
    - Best practice guidelines are available for many applications
  - How to select the kernel parameters?
    - (Massive) cross-validation.
    - Usually, several parameters are optimized together in a grid search.
  - Solving the quadratic programming problem
    - Standard QP solvers do not perform too well on SVM task.
    - Dedicated methods have been developed for this, e.g. SMO.
  - Speed of evaluation
    - Evaluating $y(x)$ scales linearly in the number of SVs.
    - Too expensive if we have a large number of support vectors.
    ⇒ There are techniques to reduce the effective SV set.
  - Training for very large datasets (millions of data points)
    - Stochastic gradient descent and other approximations can be used
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  ➢ Dual formulation
  ➢ Soft-margin classification

• Nonlinear Support Vector Machines
  ➢ Nonlinear basis functions
  ➢ The Kernel trick
  ➢ Mercer’s condition
  ➢ Popular kernels

• Analysis
  ➢ Error function

• Applications
SVM – Analysis

• Traditional soft-margin formulation

$$\min_{\mathbf{w} \in \mathbb{R}^D, \xi_n \in \mathbb{R}^+} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{n=1}^{N} \xi_n$$

subject to the constraints

$$t_n y(x_n) \geq 1 - \xi_n$$

• Different way of looking at it
  ➢ We can reformulate the constraints into the objective function.

$$\min_{\mathbf{w} \in \mathbb{R}^D} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{n=1}^{N} [1 - t_n y(x_n)]_+$$

where $$[x]_+ := \max\{0, x\}.$$
Recap: Error Functions

- Ideal misclassification error function (black)
  - This is what we want to approximate,
  - Unfortunately, it is not differentiable.
  - The gradient is zero for misclassified points.
  - We cannot minimize it by gradient descent.
Recap: Error Functions

\[ t_n \in \{-1, 1\} \]

Sensitive to outliers!

- **Squared error used in Least-Squares Classification**
  - Very popular, leads to closed-form solutions.
  - However, sensitive to outliers due to squared penalty.
  - Penalizes “too correct” data points
  - \( E(z_n) \)

Ideal misclassification error

Squared error

\[ z_n = t_n y(x_n) \]
Error Functions (Loss Functions)

- "Hinge error" used in SVMs
  - Zero error for points outside the margin ($z_n > 1$) $\Rightarrow$ sparsity
  - Linear penalty for misclassified points ($z_n < 1$) $\Rightarrow$ robustness
  - Not differentiable around $z_n = 1$ $\Rightarrow$ Cannot be optimized directly.

Ideal misclassification error

Squared error

Hinge error

Robust to outliers!

Not differentiable!

Favors sparse solutions!

$z_n = t_n y(x_n)$

Image source: Bishop, 2006
SVM – Discussion

• SVM optimization function

\[
\min_{\mathbf{w} \in \mathbb{R}^D} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{n=1}^{N} [1 - t_n y(\mathbf{x}_n)]_+
\]

- L₂ regularizer
- Hinge loss

• Hinge loss enforces sparsity
  - Only a subset of training data points actually influences the decision boundary.
  - This is different from sparsity obtained through the regularizer! There, only a subset of input dimensions are used.
  - Unconstrained optimization, but non-differentiable function.
  - Solve, e.g. by subgradient descent
  - Currently most efficient: stochastic gradient descent

Slide adapted from Christoph Lampert
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• Applications
Example Application: Text Classification

- **Problem:**
  - Classify a document in a number of categories

- **Representation:**
  - “Bag-of-words” approach
  - Histogram of word counts (on learned dictionary)
    - Very high-dimensional feature space (~10,000 dimensions)
    - Few irrelevant features

- **This was one of the first applications of SVMs**
  - T. Joachims (1997)
Example Application: Text Classification

- Results:

<table>
<thead>
<tr>
<th></th>
<th>Bayes</th>
<th>Rocchio</th>
<th>C4.5</th>
<th>k-NN</th>
<th>SVM (poly) degree $d = 1$</th>
<th>SVM (poly) degree $d = 2$</th>
<th>SVM (poly) degree $d = 3$</th>
<th>SVM (poly) degree $d = 4$</th>
<th>SVM (poly) degree $d = 5$</th>
<th>SVM (poly) degree $d = 0.6$</th>
<th>SVM (poly) degree $d = 0.8$</th>
<th>SVM (poly) degree $d = 1.0$</th>
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</tr>
</tbody>
</table>

combined: **86.0**
combined: **86.4**
Example Application: Text Classification

- This is also how you could implement a simple spam filter...

Diagram:
- Incoming email
- Dictionary
- Word activations
- SVM
- Mailbox
- Trash

B. Leibe
Example Application: OCR

- Handwritten digit recognition
  - US Postal Service Database
  - Standard benchmark task for many learning algorithms
Historical Importance

• USPS benchmark
  - 2.5% error: human performance

• Different learning algorithms
  - 16.2% error: Decision tree (C4.5)
  - 5.9% error: (best) 2-layer Neural Network
  - 5.1% error: LeNet 1 – (massively hand-tuned) 5-layer network

• Different SVMs
  - 4.0% error: Polynomial kernel (p=3, 274 support vectors)
  - 4.1% error: Gaussian kernel  (σ=0.3, 291 support vectors)
Example Application: OCR

- Results
  - Almost no overfitting with higher-degree kernels.

<table>
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<tr>
<th>degree of polynomial</th>
<th>dimensionality of feature space</th>
<th>support vectors</th>
<th>raw error</th>
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Example Application: Object Detection

- Sliding-window approach

  - E.g. histogram representation (HOG)
    - Map each grid cell in the input window to a histogram of gradient orientations.
    - Train a linear SVM using training set of pedestrian vs. non-pedestrian windows.

[Dalal & Triggs, CVPR 2005]
Example Application: Pedestrian Detection

N. Dalal, B. Triggs, Histograms of Oriented Gradients for Human Detection, CVPR 2005
Many Other Applications

- Lots of other applications in all fields of technology
  - OCR
  - Text classification
  - Computer vision
  - ...
  - High-energy physics
  - Monitoring of household appliances
  - Protein secondary structure prediction
  - Design on decision feedback equalizers (DFE) in telephony

(Detailed references in *Schoelkopf & Smola, 2002*, pp. 221)
References and Further Reading

• More information on SVMs can be found in Chapter 7.1 of Bishop’s book. You can also look at Schölkopf & Smola (some chapters available online).

  Christopher M. Bishop  
  Pattern Recognition and Machine Learning  
  Springer, 2006

  B. Schölkopf, A. Smola  
  Learning with Kernels  
  MIT Press, 2002  
  http://www.learning-with-kernels.org/

• A more in-depth introduction to SVMs is available in the following tutorial:


B. Leibe