Advanced Machine Learning
Lecture 2

Linear Regression

27.10.2016

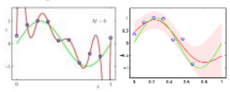
Bastian Leibe
RWTH Aachen
http://www.vision.rwth-aachen.de/

leibe@vision.rwth-aachen.de

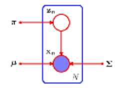
This Lecture: *Advanced Machine Learning*

$f : \mathcal{X} \to \mathbb{R}$

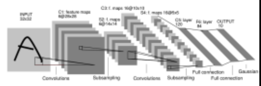
- Regression Approaches
  - Linear Regression
  - Regularization (Ridge, Lasso)
  - Gaussian Processes



- Learning with Latent Variables
  - EM and Generalizations
  - Approximate Inference



- Deep Learning
  - Neural Networks
  - CNNs, RNNs, RBMs, etc.



Topics of This Lecture

- Recap: Important Concepts from ML Lecture
  - Probability Theory
  - Bayes Decision Theory
  - Maximum Likelihood Estimation
  - Bayesian Estimation
- A Probabilistic View on Regression
  - Least-Squares Estimation as Maximum Likelihood
  - Predictive Distribution
  - Maximum-A-Posteriori (MAP) Estimation
  - Bayesian Curve Fitting
- Discussion

Recap: The Rules of Probability

- Basic rules

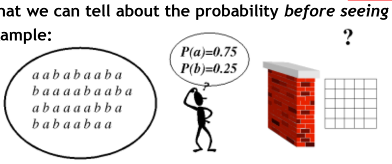
Sum Rule	$p(X) = \sum_Y p(X,Y)$
Product Rule	$p(X,Y) = p(Y\|X)p(X)$

- From those, we can derive

Bayes' Theorem	$p(Y\|X) = \frac{p(X\|Y)p(Y)}{p(X)}$
where	$p(X) = \sum_Y p(X\|Y)p(Y)$

Recap: Bayes Decision Theory

- Concept 1: Priors (a priori probabilities) $p(C_k)$
  - What we can tell about the probability *before seeing the data*.
  - Example:



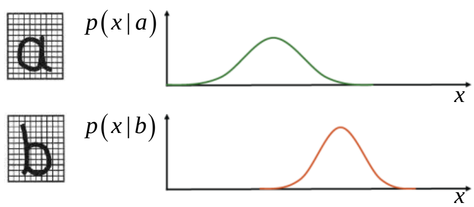
$C_1 = a$ $\quad p(C_1) = 0.75$

$C_2 = b$ $\quad p(C_2) = 0.25$

- In general: $\sum_k p(C_k) = 1$

Recap: Bayes Decision Theory

- Concept 2: Conditional probabilities $p(x|C_k)$
  - Let x be a feature vector.
  - x measures/describes certain properties of the input.
    - E.g. number of black pixels, aspect ratio, …
  - $p(x|C_k)$ describes its likelihood for class C_k.



RWTH AACHEN UNIVERSITY

Recap: Bayes Decision Theory

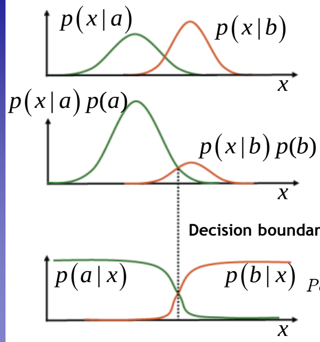
- Concept 3: Posterior probabilities $p(C_k \mid x)$
  - We are typically interested in the *a posteriori* probability, i.e. the probability of class C_k given the measurement vector x.
- Bayes' Theorem:

$$p(C_k \mid x) = \frac{p(x \mid C_k)\, p(C_k)}{p(x)} = \frac{p(x \mid C_k)\, p(C_k)}{\sum_i p(x \mid C_i)\, p(C_i)}$$

- Interpretation

$$Posterior = \frac{Likelihood \times Prior}{Normalization\ Factor}$$

Recap: Bayes Decision Theory



Likelihood

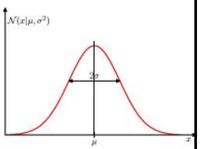
Likelihood × Prior

$$Posterior = \frac{Likelihood \times Prior}{NormalizationFactor}$$

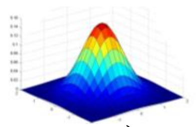
Recap: Gaussian (or Normal) Distribution

- One-dimensional case
  - Mean μ
  - Variance σ^2

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$



- Multi-dimensional case
  - Mean μ
  - Covariance Σ



$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Side Note

- Notation
  - In many situations, it will be preferable to work with the inverse of the covariance matrix $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$$

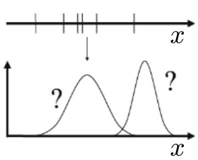
  - We call $\boldsymbol{\Lambda}$ the precision matrix.
  - We can therefore also write the Gaussian as

$$\mathcal{N}(x|\mu,\lambda^{-1}) = \frac{1}{\sqrt{2\pi}\lambda^{-1/2}} \exp\left\{-\frac{\lambda}{2}(x-\mu)^2\right\}$$

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda}^{-1}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Lambda}|^{-1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Recap: Parametric Methods

- Given
  - Data $X = \{x_1, x_2, \ldots, x_N\}$
  - Parametric form of the distribution with parameters θ
  - E.g. for Gaussian distrib.: $\theta = (\mu, \sigma)$



- Learning
  - Estimation of the parameters θ
- Likelihood of θ
  - Probability that the data X have indeed been generated from a probability density with parameters θ

$$L(\theta) = p(X|\theta)$$

Recap: Maximum Likelihood Approach

- Computation of the likelihood
  - Single data point: $p(x_n|\theta) \quad = \mathcal{N}(x_n|\mu,\sigma^2)$
  - Assumption: all data points $X = \{x_1, \ldots, x_n\}$ are independent

$$L(\theta) = p(X|\theta) = \prod_{n=1}^{N} p(x_n|\theta)$$

  - Log-likelihood

$$E(\theta) = -\ln L(\theta) = -\sum_{n=1}^{N} \ln p(x_n|\theta)$$

- Estimation of the parameters θ (Learning)
  - Maximize the likelihood (=minimize the negative log-likelihood)

  $\Rightarrow$ Take the derivative and set it to zero.

$$\frac{\partial}{\partial\theta}E(\theta) = -\sum_{n=1}^{N} \frac{\frac{\partial}{\partial\theta}p(x_n|\theta)}{p(x_n|\theta)} \stackrel{!}{=} 0$$

Recap: Maximum Likelihood Approach

- Applying ML to estimate the parameters of a Gaussian, we obtain

$$\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} x_n \qquad \text{“sample mean”}$$

- In a similar fashion, we get

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \hat{\mu})^2 \qquad \text{“sample variance”}$$

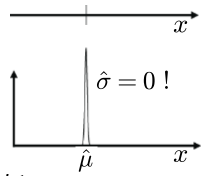
- $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$ is the **Maximum Likelihood estimate** for the parameters of a Gaussian distribution.
- This is a very important result.
- Unfortunately, it is biased…

Recap: Maximum Likelihood - Limitations

- Maximum Likelihood has several significant limitations
  - It systematically underestimates the variance of the distribution!
  - E.g. consider the case

$$N = 1, X = \{x_1\}$$

$\Rightarrow$ Maximum-likelihood estimate:



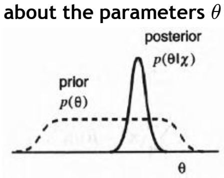
  - We say ML *overfits to the observed data*.
  - We will still often use ML, but it is important to know about this effect.

Recap: Deeper Reason

- Maximum Likelihood is a **Frequentist** concept
  - In the **Frequentist view**, probabilities are the frequencies of random, repeatable events.
  - These frequencies are fixed, but can be estimated more precisely when more data is available.
- This is in contrast to the **Bayesian** interpretation
  - In the **Bayesian view**, probabilities quantify the uncertainty about certain states or events.
  - This uncertainty can be revised in the light of new evidence.
- Bayesians and Frequentists do not like each other too well…

Recap: Bayesian Approach to Learning

- Conceptual shift
  - Maximum Likelihood views the true parameter vector θ to be unknown, but fixed.
  - In Bayesian learning, we consider θ to be a random variable.
- This allows us to use knowledge about the parameters θ
  - i.e. to use a prior for θ
  - Training data then converts this prior distribution on θ into a posterior probability density.



  - The prior thus encodes knowledge we have about the type of distribution we expect to see for θ.

Recap: Bayesian Learning Approach

- Bayesian view:
  - Consider the parameter vector θ as a random variable.
  - When estimating the parameters, what we compute is

$$p(x|X) = \int p(x, \theta|X)\,d\theta$$

Assumption: given θ, this doesn't depend on X anymore

$$p(x, \theta|X) = p(x|\theta, \cancel{X})p(\theta|X)$$

$$p(x|X) = \int \underbrace{p(x|\theta)}p(\theta|X)\,d\theta$$

This is entirely determined by the parameter θ (i.e. by the parametric form of the pdf).

Recap: Bayesian Learning Approach

$$p(x|X) = \int p(x|\theta)\underbrace{p(\theta|X)}d\theta$$

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)} = \frac{p(\theta)}{p(X)}L(\theta)$$

$$p(X) = \int p(X|\theta)p(\theta)\,d\theta = \int L(\theta)p(\theta)\,d\theta$$

- Inserting this above, we obtain

$$p(x|X) = \int \frac{p(x|\theta)L(\theta)p(\theta)}{p(X)}d\theta = \int \frac{p(x|\theta)L(\theta)p(\theta)}{\int L(\theta)p(\theta)d\theta}d\theta$$

Recap: Bayesian Learning Approach

- Discussion

$$p(x|X) = \int \frac{p(x|\theta)L(\theta)p(\theta)}{\int L(\theta)p(\theta)d\theta} d\theta$$

Estimate for x based on parametric form θ

Likelihood of the parametric form θ given the data set X.

Prior for the parameters θ

Normalization: integrate over all possible values of θ

  - The more uncertain we are about θ, the more we average over all possible parameter values.

Topics of This Lecture

- Recap: Important Concepts from ML Lecture
  - Probability Theory
  - Bayes Decision Theory
  - Maximum Likelihood Estimation
  - Bayesian Estimation
- **A Probabilistic View on Regression**
  - Least-Squares Estimation as Maximum Likelihood
  - Predictive Distribution
  - Maximum-A-Posteriori (MAP) Estimation
  - Bayesian Curve Fitting
- Discussion

Curve Fitting Revisited

- In the last lecture, we've looked at curve fitting in terms of error minimization…
- Now: View the problem from a probabilistic perspective
  - Goal is to make predictions for target variable t given new value for input variable x.
  - Basis: training set $\mathbf{x} = (x_1, \ldots, x_N)^T$ with target values $\mathbf{t} = (t_1, \ldots, t_N)^T$.
  - We express our uncertainty over the value of the target variable using a probability distribution

$$p(t|x, \mathbf{w}, \beta)$$

Probabilistic Regression

- First assumption:
  - Our target function values t are generated by adding noise to the ideal function estimate:

$$t = y(\mathbf{x}, \mathbf{w}) + \epsilon$$

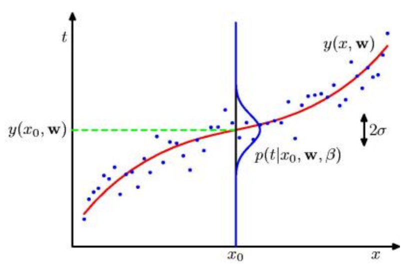
Target function value; Regression function; Input value; Weights or parameters; Noise

- Second assumption:
  - The noise is Gaussian distributed.

$$p(t|\mathbf{x}, \mathbf{w}, \beta) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$$

Mean; Variance (β precision)

Visualization: Gaussian Noise



Probabilistic Regression

- Given
  - Training data points: $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{R}^{d \times n}$
  - Associated function values: $\mathbf{t} = [t_1, \ldots, t_n]^T$
- Conditional likelihood (assuming i.i.d. data)

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta) = \prod_{n=1}^{N} \mathcal{N}(t_n|y(\mathbf{x}_n, \mathbf{w}), \beta^{-1}) = \prod_{n=1}^{N} \mathcal{N}(t_n|\underbrace{\mathbf{w}^T\phi(\mathbf{x}_n)}, \beta^{-1})$$

Generalized linear regression function

$\Rightarrow$ Maximize w.r.t. $\mathbf{w}$, β

Maximum Likelihood Regression

- Simplify the log-likelihood

$$\log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) = \sum_{n=1}^{N} \log \mathcal{N}(t_n|y(\mathbf{x}_n,\mathbf{w}),\beta^{-1})$$

$$\mathcal{N}(x|\mu,\beta^{-1}) = \frac{1}{\sqrt{2\pi}\beta^{-1/2}} \exp\left\{-\frac{\beta}{2}(x-\mu)^2\right\}$$

$$= \sum_{n=1}^{N}\left[\log\left(\frac{\sqrt{\beta}}{\sqrt{2\pi}}\right) - \frac{\beta}{2}\{y(\mathbf{x}_n,\mathbf{w})-t_n\}^2\right]$$

$$= -\frac{\beta}{2}\underbrace{\sum_{n=1}^{N}\{t_n - y(\mathbf{x}_n,\mathbf{w})\}^2}_{\text{Sum-of-squares error}} + \underbrace{\frac{N}{2}\log\beta - \frac{N}{2}\log(2\pi)}_{\text{Constants}}$$

Maximum Likelihood Regression

$$\log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) = -\frac{\beta}{2}\sum_{n=1}^{N}\{t_n - y(\mathbf{x}_n,\mathbf{w})\}^2 + \frac{N}{2}\log\beta - \frac{N}{2}\log(2\pi)$$

$$= -\frac{\beta}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^T\phi(\mathbf{x}_n)\}^2 + \frac{N}{2}\log\beta - \frac{N}{2}\log(2\pi)$$

- Gradient w.r.t. $\mathbf{w}$:

$$\nabla_{\mathbf{w}} \log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) = -\beta\sum_{n=1}^{N}(t_n - \mathbf{w}^T\phi(\mathbf{x}_n))\phi(\mathbf{x}_n)$$

Maximum Likelihood Regression

$$\nabla_{\mathbf{w}} \log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) = -\beta\sum_{n=1}^{N}(t_n - \mathbf{w}^T\phi(\mathbf{x}_n))\phi(\mathbf{x}_n)$$

- Setting the gradient to zero:

$$0 = -\beta\sum_{n=1}^{N}(t_n - \mathbf{w}^T\phi(\mathbf{x}_n))\phi(\mathbf{x}_n)$$

$$\Leftrightarrow \sum_{n=1}^{N} t_n\phi(\mathbf{x}_n) = \left[\sum_{n=1}^{N}\phi(\mathbf{x}_n)\phi(\mathbf{x}_n)^T\right]\mathbf{w}$$

$$\Leftrightarrow \mathbf{\Phi}\mathbf{t} = \mathbf{\Phi}\mathbf{\Phi}^T\mathbf{w} \qquad \mathbf{\Phi} = [\phi(\mathbf{x}_1),\ldots,\phi(\mathbf{x}_n)]$$

$$\Leftrightarrow \mathbf{w}_{\mathrm{ML}} = (\mathbf{\Phi}\mathbf{\Phi}^T)^{-1}\mathbf{\Phi}\mathbf{t}$$

← Same as in least-squares regression!

⇒ *Least-squares regression is equivalent to Maximum Likelihood under the assumption of Gaussian noise.*

Role of the Precision Parameter

- Also use ML to determine the precision parameter β:

$$\log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) = -\frac{\beta}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^T\phi(\mathbf{x}_n)\}^2 + \frac{N}{2}\log\beta - \frac{N}{2}\log(2\pi)$$

- Gradient w.r.t. β:

$$\nabla_{\beta} \log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) = -\frac{1}{2}\sum_{n=1}^{N}\{t_n - \mathbf{w}^T\phi(\mathbf{x}_n)\}^2 + \frac{N}{2}\frac{1}{\beta}$$

$$\frac{1}{\beta_{\mathrm{ML}}} = \frac{1}{N}\sum_{n=1}^{N}\{t_n - \mathbf{w}^T\phi(\mathbf{x}_n)\}^2$$

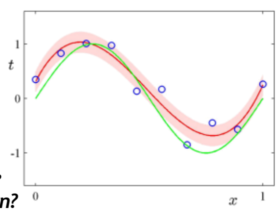
⇒ *The inverse of the noise precision is given by the residual variance of the target values around the regression function.*

Predictive Distribution

- Having determined the parameters $\mathbf{w}$ and β, we can now make predictions for new values of $\mathbf{x}$.

$$p(t|\mathbf{X},\mathbf{w}_{\mathrm{ML}},\beta_{\mathrm{ML}}) = \mathcal{N}(t|y(\mathbf{x},\mathbf{w}_{\mathrm{ML}}),\beta_{\mathrm{ML}}^{-1})$$

- This means
  - Rather than giving a point estimate, we can now also give an estimate of the estimation uncertainty.



- *What else can we do in the Bayesian view of regression?*

Image source: C.M. Bishop, 2006

MAP: A Step Towards Bayesian Estimation…

- Introduce a prior distribution over the coefficients $\mathbf{w}$.
  - For simplicity, assume a zero-mean Gaussian distribution

$$p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|\mathbf{0},\alpha^{-1}\mathbf{I}) = \left(\frac{\alpha}{2\pi}\right)^{(M+1)/2}\exp\left\{-\frac{\alpha}{2}\mathbf{w}^T\mathbf{w}\right\}$$

  - New hyperparameter α controls the distribution of model parameters.

- Express the posterior distribution over $\mathbf{w}$.
  - Using Bayes' theorem:

$$p(\mathbf{w}|\mathbf{X},\mathbf{t},\beta,\alpha) \propto p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta)p(\mathbf{w}|\alpha)$$

  - We can now determine $\mathbf{w}$ by maximizing the posterior.
  - This technique is called maximum-a-posteriori (MAP).

MAP Solution

- Minimize the negative logarithm

$$-\log p(\mathbf{w}|\mathbf{X},\mathbf{t},\beta,\alpha) \propto -\log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) - \log p(\mathbf{w}|\alpha)$$

$$-\log p(\mathbf{t}|\mathbf{X},\mathbf{w},\beta) = \frac{\beta}{2}\sum_{n=1}^{N}\{y(\mathbf{x}_n,\mathbf{w}) - t_n\}^2 + \text{const}$$

$$-\log p(\mathbf{w}|\alpha) = \frac{\alpha}{2}\mathbf{w}^T\mathbf{w} + \text{const}$$

- The MAP solution is therefore the solution of

$$\frac{\beta}{2}\sum_{n=1}^{N}\{y(\mathbf{x}_n,\mathbf{w}) - t_n\}^2 + \frac{\alpha}{2}\mathbf{w}^T\mathbf{w}$$

⇒ ***Maximizing the posterior distribution is equivalent to minimizing the regularized sum-of-squares error** (with $\lambda = \frac{\alpha}{\beta}$).*

Results of Probabilistic View on Regression

- Better understanding what linear regression *means*
  - *Least-squares regression is equivalent to ML estimation under the assumption of Gaussian noise.*
  - ⇒ We can use the predictive distribution to give an uncertainty estimate on the prediction.
  - ⇒ But: known problem with ML that it tends towards overfitting.
  - *L2-regularized regression (Ridge regression) is equivalent to MAP estimation with a Gaussian prior on the parameters w.*
  - ⇒ The prior controls the parameter values to reduce overfitting.
  - ⇒ This gives us a tool to explore more general priors.
- But still no full Bayesian Estimation yet
  - Should integrate over all values of w instead of just making a point estimate.

Bayesian Curve Fitting

- Given
  - Training data points: $\mathbf{X} = [\mathbf{x}_1,\ldots,\mathbf{x}_n] \in \mathbb{R}^{d\times n}$
  - Associated function values: $\mathbf{t} = [t_1,\ldots,t_n]^T$
  - Our goal is to predict the value of t for a new point $\mathbf{x}$.
- Evaluate the predictive distribution

$$p(t|x,\mathbf{X},\mathbf{t}) = \int p(t|x,\mathbf{w})p(\mathbf{w}|\mathbf{X},\mathbf{t})d\mathbf{w}$$

What we just computed for MAP

  - Noise distribition - again assume a Gaussian here

$$p(t|x,\mathbf{w}) = \mathcal{N}(t|y(\mathbf{x},\mathbf{w}),\beta^{-1})$$

  - Assume that parameters α and β are fixed and known for now.

Bayesian Curve Fitting

- Under those assumptions, the posterior distribution is a Gaussian and can be evaluated analytically:

$$p(t|x,\mathbf{X},\mathbf{t}) = \mathcal{N}(t|m(x),s^2(x))$$

  - where the mean and variance are given by

$$m(x) = \beta\phi(x)^T\mathbf{S}\sum_{n=1}^{N}\phi(\mathbf{x}_n)t_n$$

$$s(x)^2 = \beta^{-1} + \phi(x)^T\mathbf{S}\phi(x)$$

  - and S is the regularized covariance matrix

$$\mathbf{S}^{-1} = \alpha\mathbf{I} + \beta\sum_{n=1}^{N}\phi(\mathbf{x}_n)\phi(\mathbf{x}_n)^T$$

Analyzing the result

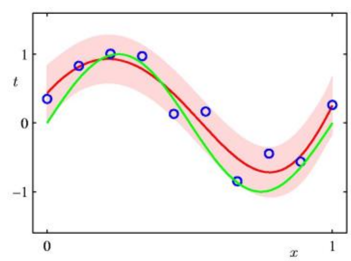
- Analyzing the variance of the predictive distribution

$$s(x)^2 = \beta^{-1} + \phi(x)^T\mathbf{S}\phi(x)$$

Uncertainty in the predicted value due to noise on the target variables (expressed already in ML)

Uncertainty in the parameters w (consequence of Bayesian treatment)

Bayesian Predictive Distribution



- Important difference to previous example
  - Uncertainty may vary with test point x!

Image source: C.M. Bishop, 2006

Topics of This Lecture

- Recap: Important Concepts from ML Lecture
  - Probability Theory
  - Bayes Decision Theory
  - Maximum Likelihood Estimation
  - Bayesian Estimation
- A Probabilistic View on Regression
  - Least-Squares Estimation as Maximum Likelihood
  - Predictive Distribution
  - Maximum-A-Posteriori (MAP) Estimation
  - Bayesian Curve Fitting
- **Discussion**

Discussion

- We now have a better understanding of regression
  - Least-squares regression: Assumption of Gaussian noise
  - ⇒ We can now also plug in different noise models and explore how they affect the error function.
  - L2 regularization as a Gaussian prior on parameters $\mathbf{w}$.
  - ⇒ We can now also use different regularizers and explore what they mean.
  - ⇒ Next lecture…
  - General formulation with basis functions $\phi(\mathbf{x})$.
  - ⇒ We can now also use different basis functions.

Discussion

- General regression formulation
  - In principle, we can perform regression in arbitrary spaces and with many different types of basis functions
  - However, there is a caveat… Can you see what it is?
- Example: Polynomial curve fitting, $M = 3$

$$y(\mathbf{x}, \mathbf{w}) = w_0 + \sum_{i=1}^{D} w_i x_i + \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij} x_i x_j + \sum_{i=1}^{D}\sum_{j=1}^{D}\sum_{k=1}^{D} w_{ijk} x_i x_j x_k$$

  - ⇒ Number of coefficients grows with D^M!
  - ⇒ The approach becomes quickly unpractical for high dimensions.
  - This is known as the curse of dimensionality.
  - We will encounter some ways to deal with this later.

References and Further Reading

- More information on linear regression can be found in Chapters 1.2.5-1.2.6 and 3.1-3.1.4 of

Christopher M. Bishop
Pattern Recognition and Machine Learning
Springer, 2006